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Journal of Approximation Theory 136 (2005) 198–229

Journal of Approximation Theory

www.elsevier.com/locate/jat

Approximation of vectors fields by thin plate splines with tension

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Received 17 December 2003; accepted 6 July 2005

Communicated by Robert Schaback

Abstract

We study a vectorial approximation problem based on thin plate splines with tension involving two positive parameters: one for the control of the oscillations and the other for the control of the divergence and rotational components of the field. The existence and uniqueness of the solution are proved and the solution is explicitly given. As special cases, we study the limit problems as the parameter controlling the divergence and the rotation converges to zero or infinity. The divergence-free and the rotation-free approximation problems are also considered. The convergence in Sobolev space is studied.

MSC: 41A65; 65D05; 65D07; 60E05

Keywords: Approximation theory; Thin plate splines; Splines with tension; Smoothing and interpolating problems; Vectorial splines; Divergence-free; Rotation-free; Sobolev space

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doi:10.1016/j.jat.2005.07.005

1. Introduction

Vector field approximation is a problem involving the reconstruction of physical fields from a set of scattered observed data. The problem can arise in many scientific applications, such as meteorological analysis, electromagnetic and fluid mechanics. In [2,3], Amodei and Benbourhim introduced a family of spline minimization problems for two-dimensional vector fields, defined by

$$\begin{cases} \mathrm{Min}\Big(\rho \int_{\mathbb{R}^2} \|\nabla \mathrm{div}(u)\|^2 dx + \int_{\mathbb{R}^2} \|\nabla \mathrm{rot}(u)\|^2 dx\Big) \\ u \in \mathcal{V}(\mathbb{R}^2;\mathbb{R}^2) \quad \text{and} \quad u(x_i) = u_i, \quad i = 1,\ldots,N, \end{cases} \tag{1.1}$$

where $\mathcal{V}(\mathbb{R}^2;\mathbb{R}^2) = D^{-2}L^2(\mathbb{R}^2) \times D^{-2}L^2(\mathbb{R}^2)$ and $D^{-2}L^2(\mathbb{R}^2)$ is the Beppo–Levi space of distributions whose derivatives of order 2 are square integrable over $\mathbb{R}^2$ (see [9]). The interpolating points are $x_i = (x_{1,i}, x_{2,i}) \in \mathbb{R}^2$ and $u_i = (u_{1,i}, u_{2,i}) \in \mathbb{R}^2$ are the data vectors. The coefficient ρ is a real positive parameter controlling the relative weight on the gradient of the divergence and the rotational fields in two-dimensional space. It is shown in [2,3] that problem (1.1) admits a unique solution.

In this work, we present a formulation for a three-dimensional vector spline approximation based on thin plate splines under tension, as introduced by Bouhamidi and Le Méhauté [5,6]. The main idea here is to introduce a vector spline depending on a tension parameter, which may be selected to avoid some extraneous inflections and oscillations.

The space $\mathcal{V}(\mathbb{R}^2;\mathbb{R}^2)$ given in problem (1.1) is modified and here is the space

$$\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3) = X^m(\mathbb{R}^3) \times X^m(\mathbb{R}^3) \times X^m(\mathbb{R}^3),$$

where $X^m(\mathbb{R}^3)$ is the space of distributions whose derivatives of order m and $m+1$ are square integrable over the three-dimensional space $\mathbb{R}^3$ [5,6].

We recall the following classical notations: let $\mathcal{D}(\mathbb{R}^3)$ denote the space of compactly supported and infinitely differentiable functions on $\mathbb{R}^3$ and $\mathcal{D}'(\mathbb{R}^3)$ the space of distributions on $\mathbb{R}^3$. Let $\mathcal{C}(\mathbb{R}^3)$ be the space of continuous functions on $\mathbb{R}^3$ and $\mathcal{C}'(\mathbb{R}^3)$ its topological dual, the space of compactly supported Radon measures on $\mathbb{R}^3$. For an integer $m > 0$, we denote by $\mathbb{P}_{m-1}(\mathbb{R}^3)$ the space of all polynomials defined over $\mathbb{R}^3$ of total degree at most equal to $m-1$. As usual, $L^2(\mathbb{R}^3)$ denotes the classical space of measurable functions which are square integrable on $\mathbb{R}^3$. The corresponding vectorial spaces are, respectively, denoted by

$$\begin{aligned}
&\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3) = \mathcal{D}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3) \times \mathcal{D}(\mathbb{R}^3),\\
&\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3) = \mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3) \times \mathcal{D}'(\mathbb{R}^3),\\
&\mathcal{C}(\mathbb{R}^3;\mathbb{R}^3) = \mathcal{C}(\mathbb{R}^3) \times \mathcal{C}(\mathbb{R}^3) \times \mathcal{C}(\mathbb{R}^3), \quad \mathcal{C}'(\mathbb{R}^3;\mathbb{R}^3) = \mathcal{C}'(\mathbb{R}^3) \times \mathcal{C}'(\mathbb{R}^3) \times \mathcal{C}'(\mathbb{R}^3),\\
&\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3) = \mathbb{P}_{m-1}(\mathbb{R}^3) \times \mathbb{P}_{m-1}(\mathbb{R}^3) \times \mathbb{P}_{m-1}(\mathbb{R}^3).
\end{aligned}$$

If T is a distribution in $\mathcal{D}'(\mathbb{R}^3)$ and φ is a test function, the action of T on φ is denoted by $\langle T, \varphi\rangle$ and if $T = (T_1, T_2, T_3)$ is a vectorial distribution in $\mathcal{D}'(\mathbb{R}^3;\mathbb{R}^3)$ and

$\varphi = (\varphi_1, \varphi_2, \varphi_3)$ is a vectorial test function, the action of T on φ is also denoted by $\langle T, \varphi \rangle$ and we have $\langle T, \varphi \rangle = \sum_{i=1}^{3} \langle T_i, \varphi_i \rangle$. If T and φ are such that the convolution products $\varphi_i * T_i$ have a sense for $i = 1, 2, 3$, then we denote by $\varphi * T$ the vectorial convolution product $\varphi * T = (\varphi_1 * T_1, \varphi_2 * T_2, \varphi_3 * T_3)$. For a matrix-valued function $F = (F_{i,j})_{\substack{1 \leqslant i \leqslant 3 \\ 1 \leqslant j \leqslant 3}}$ such that $F_{i,j} * T_j$ have a sense for $i, j = 1, 2, 3$, the convolution product $F * T$ is the vector function given by $F * T = (G_1, G_2, G_3)$ where $G_i = \sum_{j=1}^{3} F_{i,j} * T_j$ for $i = 1, 2, 3$. The notation ∂_i denotes the partial derivative $\dfrac{\partial}{\partial x_i}$ for $i = 1, 2, 3$ and the notations D^α or ∂^α denote the partial derivative of order α. The gradient of a scalar distribution T is the vectorial distribution $\nabla T = (\partial_1 T, \partial_2 T, \partial_3 T)$. The divergence, the rotational and the Laplacian of a vectorial distribution $T = (T_1, T_2, T_3)$ are given by $div\, T = \partial_1 T_1 + \partial_2 T_2 + \partial_3 T_3$, $rot\, T = (\partial_2 T_3 - \partial_3 T_2, \partial_3 T_1 - \partial_1 T_3, \partial_1 T_2 - \partial_2 T_1)$ and $\Delta T = (\Delta T_1, \Delta T_2, \Delta T_3)$, respectively. We denote by $(rot\, T)_i$ for $i = 1, 2, 3$ the ith component of $rot\, T$. We also recall the formulas $rot(rot\, T) = \nabla(div\, T) - \Delta T$, $rot(\nabla T) = 0$ and $div(rot\, T) = 0$.

The letter C refers to an arbitrary constant, and it may have different values in different contexts.

The outline of this paper is as follows: in Section 2, we recall some fundamental results and propositions involving the scalar functional space $X^m(\mathbb{R}^3)$ and we give some propositions related to the vectorial functional space $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$. In Section 3, we present the vectorial approximation problem related to thin plate splines under tension and we prove the existence and uniqueness of the solution. The solution is explicitly given with some of its properties. In Section 4, we study four particular vectorial spline problems. The first and second problems are the limit problems when the parameter, controlling the divergence and the rotational components of the vector field, goes to zero or infinity, respectively. The third and fourth problems are the divergence-free problem and the rotation-free problem, respectively. In each case, we prove the existence and uniqueness of the solution and we give an explicit representation of the solution together with some of its properties. We study the convergence to the limit problems. In Section 5, we study the convergence in the classical vectorial Sobolev space, on a bounded open set Ω of $\mathbb{R}^3$.

2. Functional spaces

2.1. Scalar functional space

We recall some notations and properties given in [5,6]. Let $m \geqslant 2$ be a given integer. We consider the space

$$X^m(\mathbb{R}^3) = \left\{ f \in \mathcal{D}'(\mathbb{R}^3) : \forall \alpha \in \mathbb{N}^3, |\alpha| = m, m+1,\ D^\alpha f \in L^2(\mathbb{R}^3) \right\}. \tag{2.1}$$

Let τ be a positive real parameter, called the parameter of tension. Let x be a generic element of $\mathbb{R}^3$ and let dx denote Lebesgue's measure on $\mathbb{R}^3$. In the space $X^m(\mathbb{R}^3)$, we consider the semi-scalar product

$$(f|g)_{X^m} = \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \int_{\mathbb{R}^3} D^\alpha f(x)\, D^\alpha g(x)\, dx + \tau^2 \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^3} D^\alpha f(x)\, D^\alpha g(x)\, dx. \tag{2.2}$$

The associated semi-norm of (2.2) is denoted by $|\,.\,|_{X^m}$. The space $X^m(\mathbb{R}^3)$ endowed with the semi-inner product $(\,.|.\,)_{X^m}$ is a semi-Hilbert space with the null subspace $\mathbb{P}_{m-1}(\mathbb{R}^3)$. We note that the elements of $X^m(\mathbb{R}^3)$ are tempered distributions.

For any element f of $X^m(\mathbb{R}^3)$ and for any function φ of $\mathcal{D}(\mathbb{R}^3)$, we have the following equalities:

$$(f|\varphi)_{X^m} = \langle \Delta_{m,\tau} f, \varphi \rangle = \langle f, \Delta_{m,\tau}\varphi \rangle,$$

where the operator $\Delta_{m,\tau}$ is defined by

$$\Delta_{m,\tau} = (-1)^{m+1} \Delta^m \Big[\Delta - \tau^2 I \Big]. \tag{2.3}$$

The operator Δ^m is the mth iterate of the usual Laplace operator and I stands for the identity operator. A fundamental solution of the differential operator $\Delta_{m,\tau}$ is a tempered distribution E_m on $\mathbb{R}^3$ satisfying the relation

$$\Delta_{m,\tau} E_m = (-1)^{m+1} (\Delta^{m+1} - \tau^2 \Delta^m) E_m = \delta, \tag{2.4}$$

where δ is the Dirac measure at the origin. An expression of a fundamental solution of the differential operator $\Delta_{m,\tau}$ is given in [6] by

$$E_m(x) = \frac{(-1)^m}{4\pi\tau^{2m}||x||} \left(e^{-\tau||x||} - \sum_{k=0}^{2m-2} \frac{(-\tau||x||)^k}{k!} \right), \tag{2.5}$$

where $||x||$ is the Euclidean norm of the vector x of $\mathbb{R}^3$.

The Fourier transform $\widehat{E}_m$ of the fundamental solution E_m of the operator $\Delta_{m,\tau}$ satisfies the relation

$$\widehat{E}_m(\xi) = \frac{1}{(||2\pi\xi||^2 + \tau^2)} \left[C_1 F_p\left(\frac{1}{||2\pi\xi||^{2m}} \right) + \sum_{|\alpha| \leqslant 2m-1} C_\alpha D^\alpha \delta \right], \tag{2.6}$$

where C_1 and C_α are given constants and F_p is the symbol of the finite part.

Let μ be a compactly supported measure. We say that μ is orthogonal to $\mathbb{P}_{m-1}(\mathbb{R}^3)$ if $\langle \mu, p \rangle = \int_{\mathbb{R}^3} p(x) d\mu(x) = 0$ for all $p \in \mathbb{P}_{m-1}(\mathbb{R}^3)$.

Proposition 2.1. *For all compactly supported measures μ orthogonal to the space $\mathbb{P}_{m-1}(\mathbb{R}^3)$, we have $\partial^\alpha(E_{m+1} * \mu) \in X^m(\mathbb{R}^3)$ for all multi-indices α such that $2 \leqslant |\alpha| \leqslant m+1$.*

Proof. We will prove that the Fourier transform of $\partial^\alpha(E_{m+1} * \mu)$ is square integrable on $\mathbb{R}^3$ for all multi-indices α such that $m+2 \leqslant |\alpha| \leqslant 2m+2$. The Fourier transform of $\partial^\alpha(E_{m+1} * \mu)$

is the product of the distribution $\widehat{E}_{m+1}$ with the $\mathcal{C}^\infty$ function $\xi \mapsto (2i\pi\xi)^\alpha \widehat{\mu}(\xi)$. The product of $D^\beta\delta$ by the function $\xi \mapsto (2i\pi\xi)^\alpha \widehat{\mu}(\xi)$ is zero for $|\beta| \leqslant |\alpha| + m - 1$ which is satisfied for $|\beta| \leqslant 2m+1$ and $|\alpha| \geqslant m+2$. From relation (2.6) we obtain that the Fourier transform of $\partial^\alpha(E_{m+1} * \mu)$ is given by

$$\mathcal{F}[\partial^\alpha(\mu * E_{m+1})] = C\,\frac{(2i\pi\xi)^\alpha\,\widehat{\mu}(\xi)}{(||2\pi\xi||^2 + \tau^2)} F_p\Big(\frac{1}{||2\pi\xi||^{2(m+1)}}\Big). \tag{2.7}$$

As μ is orthogonal to $\mathbb{P}_{m-1}(\mathbb{R}^3)$, then all the derivatives of order $\leqslant m-1$ of the Fourier transform $\widehat{\mu}$ of μ disappears at the origin. Thus there is a positive constant C such that in the neighborhood of the origin, we have $|\widehat{\mu}(\xi)| \leqslant C||\xi||^m$. Then, in the neighborhood $\mathcal{N}$ of the origin, we have the estimation

$$\int_{\mathcal{N}} \frac{||(2i\pi\xi)^\alpha\,\widehat{\mu}(\xi)||^2}{||2\pi\xi||^{4(m+1)}(||2\pi\xi||^2+\tau^2)^2}\,d\xi \leqslant C\int_{\mathcal{N}} ||\xi||^{-2m+2|\alpha|-4}\,d\xi. \tag{2.8}$$

The last integral on the right-hand side of (2.8) is finite, because $2m - 2|\alpha| + 4 < 3$, i.e., $|\alpha| > m + 1/2$. The Fourier transform of $\partial^\alpha(E_{m+1} * \mu)$ is in fact a function almost everywhere equal to the measurable function $\xi \longmapsto C_1\,\dfrac{(2i\pi\xi)^\alpha\,\widehat{\mu}(\xi)}{(||2\pi\xi||^2+\tau^2)||2\pi\xi||^{2(m+1)}}$ and the finite part symbol F_p in (2.7) is useless.

The Fourier transform of a compactly supported measure is a bounded function; thus, outside a neighborhood of the origin, we have

$$\int_{\mathbb{R}^3\setminus\mathcal{N}} \frac{||(2i\pi\xi)^\alpha\,\widehat{\mu}(\xi)||^2}{||2\pi\xi||^{4(m+1)}(||2\pi\xi||^2+\tau^2)^2}\,d\xi \leqslant C\int_{\mathbb{R}^3\setminus\mathcal{N}} ||\xi||^{2|\alpha|-4m-8}\,d\xi. \tag{2.9}$$

Then, in the outside a neighborhood of the origin, the function defined by

$$\xi \longmapsto \frac{||(2i\pi\xi)^\alpha\,\widehat{\mu}(\xi)||}{||2\pi\xi||^{2(m+1)}(||2\pi\xi||^2+\tau^2)}$$

is square integrable because $4m + 8 - 2|\alpha| > 3$ i.e., $|\alpha| < 2m + 5/2$. □

Let $d(m) = \binom{m+2}{3}$ denote the dimension of $\mathbb{P}_{m-1}(\mathbb{R}^3)$ and

$$\mathcal{A} = \{x_i = (x_{1,i}, x_{2,i}, x_{3,i}) \in \mathbb{R}^3, \quad i = 1, \ldots, N\}$$

a given ordered set of $N > d(m)$ distinct points of $\mathbb{R}^3$ which contains a $\mathbb{P}_{m-1}(\mathbb{R}^3)$-unisolvent subset of $d(m)$ points (for convenience, we assume this is the subset of the first $d(m)$ points). The $\mathbb{P}_{m-1}(\mathbb{R}^3)$-unisolvence condition is equivalent to the existence of a basis $(p_j)_{1\leqslant j\leqslant d(m)}$ of $\mathbb{P}_{m-1}(\mathbb{R}^3)$ such that

$$p_j(x_i) = \delta_{ij},\ i, j = 1, \ldots, d(m) \tag{2.10}$$

where δ_{ij} is the Kronecker symbol. Let $r = \min_{1 \leqslant i \neq j \leqslant N} ||x_i - x_j||/2$ and consider the functions $\phi_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$ for $i = 1, \ldots, N$ defined by

$$\phi_i(x) = \begin{cases} \exp(1)\exp(-\dfrac{r^2}{r^2 - ||x - x_i||^2}) & \text{if } ||x - x_i|| < r, \\ 0 & \text{otherwise.} \end{cases} \tag{2.11}$$

The functions $\phi_1, \ldots, \phi_N$ are in $\mathcal{D}(\mathbb{R}^3)$ and satisfy the following conditions:

$$\phi_j(x_i) = \delta_{ij},\ i, j = 1, \ldots, N. \tag{2.12}$$

Now we consider the bilinear form on $X^m(\mathbb{R}^3)$ defined by

$$\langle u|v\rangle_{X^m} = (u|v)_{X^m} + \sum_{i=1}^{d(m)} u(x_i)v(x_i), \tag{2.13}$$

with its associated quadratic form denoted $||\,.\,||_{X^m}$.

Proposition 2.2. *The space $(X^m(\mathbb{R}^3), \langle\,.\,|\,.\,\rangle_{X^m})$ is a Hilbert space continuously embedded in $\mathcal{C}(\mathbb{R}^3)$ and $\mathbb{P}_{m-1}(\mathbb{R}^3) + \mathcal{D}(\mathbb{R}^3)$ is dense in $X^m(\mathbb{R}^3)$.*

Proof. (see [6]). □

2.2. Vectorial functional space

Now we define the product space $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ as

$$\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3) = X^m(\mathbb{R}^3) \times X^m(\mathbb{R}^3) \times X^m(\mathbb{R}^3), \tag{2.14}$$

possessing the scalar product

$$\langle u|v\rangle_{\mathcal{V}^m} = \sum_{i=1}^{3} \langle u_i|v_i\rangle_{X^m}, \tag{2.15}$$

where $u = (u_1, u_2, u_3)$, $v = (v_1, v_2, v_3) \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$. The associated norm of (2.15) is denoted by $||\,.\,||_{\mathcal{V}^m}$. The next proposition can be viewed as an immediate consequence of Proposition 2.2.

Proposition 2.3. *The space $(\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3), \langle\,.\,|\,.\,\rangle_{\mathcal{V}^m})$ is a Hilbert space, continuously embedded in $\mathcal{C}(\mathbb{R}^3; \mathbb{R}^3)$ and $\mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3) + \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ is dense in $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$.*

Let $\rho >$ be a positive parameter. We consider the bilinear (and quadratic) forms $D_{m,\tau}$, $R_{m,\tau}$ and $J_{m,\tau,\rho}$ defined on $\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$, for all u and $v \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$, as follows:

$$\begin{aligned} &D_{m,\tau}(u,v) = (\mathit{div}\,u|\mathit{div}\,v)_{X^{m-1}}, \quad D_{m,\tau}(u) = D_{m,\tau}(u,u),\\ &R_{m,\tau}(u,v) = \sum_{i=1}^{3}((\mathit{rot}\,u)_i|(\mathit{rot}\,v)_i)_{X^{m-1}}, \quad R_{m,\tau}(u) = R_{m,\tau}(u,u),\\ &J_{m,\tau,\rho}(u,v) = \rho D_{m,\tau}(u,v) + R_{m,\tau}(u,v), \quad J_{m,\tau,\rho}(u) = J_{m,\tau,\rho}(u,u), \end{aligned} \tag{2.16}$$

where τ and $(.\,|\,.)_{X^{m-1}}$ are given in (2.2).

Proposition 2.4. *For all $u = (u_1,u_2,u_3)$ and $v = (v_1,v_2,v_3) \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$, we have the relation $D_{m,\tau}(u,v) + R_{m,\tau}(u,v) = \sum_{i=1}^{3}(u_i|v_i)_{X^m}$.*

Proof. The terms $\int_{\mathbb{R}^3} D^\alpha u_k(\xi) D^\beta v_l(\xi)\,d\xi$ of $D_{m,\tau}$ cancel each other out with the corresponding terms $\int_{\mathbb{R}^3} D^{\alpha'} u_k(\xi) D^{\beta'} v_l(\xi)\,d\xi$ of $R_{m,\tau}$ for α, α', β and β' such that $\alpha - \alpha' = \beta' - \beta = e_i - e_j$ with $l \neq k$, $i \neq j$ and e_i is the ith unit vector of the canonical basis of $\mathbb{R}^3$. □

Let us consider the following spaces $\mathcal{Y}_\alpha$, $\mathcal{Y}_m$, $\mathcal{Y}_D$, $\mathcal{Y}_R$ and $\mathcal{Y}$ given by

$$\begin{aligned} &\mathcal{Y}_\alpha = L^2(\mathbb{R}^3) \quad \forall \alpha : |\alpha| = m, m-1, \quad \mathcal{Y}_m = \prod_{|\alpha|=m,m-1} \mathcal{Y}_\alpha,\\ &\mathcal{Y}_D = \mathcal{Y}_m, \quad \mathcal{Y}_R = \mathcal{Y}_m \times \mathcal{Y}_m \times \mathcal{Y}_m, \quad \mathcal{Y} = \mathcal{Y}_D \times \mathcal{Y}_R. \end{aligned} \tag{2.17}$$

Then $f = (f_D, f_R)$ denotes an element of $\mathcal{Y} = \mathcal{Y}_D \times \mathcal{Y}_R$ where

$$\begin{aligned} &f_D = (f_{D,\alpha})_{|\alpha|=m,m-1} \in \mathcal{Y}_D,\\ &f_R = (f_{R_{1,\alpha}}, f_{R_{2,\alpha}}, f_{R_{3,\alpha}})_{|\alpha|=m,m-1} \in \mathcal{Y}_R. \end{aligned} \tag{2.18}$$

The spaces $\mathcal{Y}_D$, $\mathcal{Y}_R$ and $\mathcal{Y}$, respectively, possess the following scalar products:

$$\begin{aligned} \langle f_D|g_D\rangle_{\mathcal{Y}_D} = &\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^3} f_{D,\alpha}(x) g_{D,\alpha}(x)dx\\ &+\tau^2 \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \int_{\mathbb{R}^3} f_{D,\alpha}(x) g_{D,\alpha}(x)dx, \end{aligned} \tag{2.19}$$

$$\begin{aligned} \langle f_R|g_R\rangle_{\mathcal{Y}_R} = \sum_{i=1}^{3} \Bigg[&\sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_{\mathbb{R}^3} f_{R_{i,\alpha}}(x) g_{R_{i,\alpha}}(x)dx\\ &+ \tau^2 \sum_{|\alpha|=m-1} \frac{(m-1)!}{\alpha!} \int_{\mathbb{R}^3} f_{R_{i,\alpha}}(x) g_{R_{i,\alpha}}(x)dx \Bigg], \end{aligned} \tag{2.20}$$

and

$$\langle f|g\rangle_{\mathcal{Y}} = \rho\langle f_D|g_D\rangle_{\mathcal{Y}_D} + \langle f_R|g_R\rangle_{\mathcal{Y}_R}. \tag{2.21}$$

The associated norms are denoted $||\,.\,||_{\mathcal{Y}_D}$, $||\,.\,||_{\mathcal{Y}_R}$ and $||\,.\,||_{\mathcal{Y}}$, respectively. We note that the spaces $\mathcal{Y}_D$, $\mathcal{Y}_R$ and $\mathcal{Y}$ possessing the scalar products defined above are Hilbert spaces. We consider the linear mappings

$$T_D : \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3) \to \mathcal{Y}_D,\ T_R : \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3) \to \mathcal{Y}_R,\ T : \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3) \to \mathcal{Y} \tag{2.22}$$

defined by

$$\begin{aligned} T_D u &= (D^\alpha(\mathrm{div}\, u))_{|\alpha|=m,m-1}, \\ T_R u &= (D^\alpha(\mathrm{rot}\, u)_1, D^\alpha(\mathrm{rot}\, u)_2, D^\alpha(\mathrm{rot}\, u)_3)_{|\alpha|=m,m-1}, \\ T u &= (T_D u, T_R u). \end{aligned} \tag{2.23}$$

We point out that we have, for all $u, v \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$, the following relations

$$\begin{aligned} \langle T_D u|T_D v\rangle_{\mathcal{Y}_D} &= D_{m,\tau}(u,v), \\ \langle T_R u|T_R v\rangle_{\mathcal{Y}_R} &= R_{m,\tau}(u,v), \\ \langle T u|T v\rangle_{\mathcal{Y}} &= \rho D_{m,\tau}(u,v) + R_{m,\tau}(u,v) = J_{m,\tau,\rho}(u,v). \end{aligned} \tag{2.24}$$

Finally, let $a : X^m(\mathbb{R}^3) \to \mathbb{R}^N$ and $A : \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3) \to \mathbb{R}^{3N}$ be operators such that, $\forall f \in X^m(\mathbb{R}^3)$ and $\forall v = (v_1, v_2, v_3) \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$

$$\begin{aligned} af &= (f(x_1), \ldots, f(x_N)), \\ Av &= (v_1(x_1), \ldots, v_1(x_N), v_2(x_1), \ldots, v_2(x_N), v_3(x_1), \ldots, v_3(x_N)). \end{aligned} \tag{2.25}$$

We state the following fundamental proposition, which gives some properties of the operators *A* and *T*.

Proposition 2.5. (1) *The operators A and T are continuous from* $(\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3), ||\,.\,||_{\mathcal{V}^m})$ *to* $(\mathbb{R}^{3N}, ||\,.\,||_{\mathbb{R}^{3N}})$ *and to* $(\mathcal{Y}, ||\,.\,||_{\mathcal{Y}})$, *respectively.*

(2) $Ker(T) = \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ *and* $Ker(T) \cap Ker(A) = \{0\}$.

(3) $Ker(T) + Ker(A)$ *is a closed subspace of* $(\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3), ||\,.\,||_{\mathcal{V}^m})$.

(4) *For all* $Z \in \mathbb{R}^{3N}$ *there exist* $\Psi_Z \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ *and* $\Phi_Z \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$, *such that*:

$$A\Psi_Z = Z,\ rot(\Psi_Z) = 0,\ A\Phi_Z = Z \text{ and } div(\Phi_Z) = 0.$$

(5) *A is surjective.*

(6) $T(\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3))$ *is a closed subspace of* $(\mathcal{Y}, ||\,.\,||_{\mathcal{Y}})$.

(7) *For every closed vector subspace* $\mathcal{W}$ *of* $\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$, *the vector subspace* $T(\mathcal{W})$ *is closed in* $(\mathcal{Y}, ||\,.\,||_{\mathcal{Y}})$.

Proof. (1) From Proposition 2.4 and relation (2.24), we obviously obtain that $||Tu||_{\mathcal{Y}} \leqslant \sqrt{max(\rho,1)}\,||u||_{\mathcal{V}^m}$ for all $u \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$; this proves the continuity of *T*. The continuity of A follows from the continuous embedding of $\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$ in $\mathcal{C}(\mathbb{R}^3;\mathbb{R}^3)$ (see Proposition 2.3).

(2) For $u=(u_1,u_2,u_3)\in Ker(T)$ we have $\|Tu\|_{\mathcal{Y}}=0$. From (2.24) we obtain that $D_{m,\tau}(u,u)=R_{m,\tau}(u,u)=0$. And from Proposition 2.4 we obtain that $\sum_{i=1}^{3}|u_i|^2_{X^m}=0$. Then $u_i\in\mathbb{P}_{m-1}(\mathbb{R}^3)$ for $i=1,2,3$ namely $u\in\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$.

Let $u=(u_1,u_2,u_3)\in Ker(T)\cap Ker(A)$; then u belongs to $\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ and $u_i(x_j)=0$ for $i=1,2,3$ and $j=1,\ldots,N$. The $\mathbb{P}_{m-1}(\mathbb{R}^3)$-unisolvence of $\{x_1,\ldots,x_{d(m)}\}$ implies that $u_i=0$ for $i=1,2,3$.

(3) Is a consequence of item 2.

(4) For all $Z=(Z_i)_{1\leqslant i\leqslant N}$ with $Z_i=(z_{i,k})_{1\leqslant k\leqslant 3}\in\mathbb{R}^3$ for $1\leqslant i\leqslant N$ and $x=(t_1,t_2,t_3)$, we define the functions θ_i and φ_i by $\theta_i(x)=\sum_{k=1}^{3}z_{i,k}t_k$ and $\varphi_i(x)=\theta_i(x)\phi_i(x)$, where the functions ϕ_i are given by (2.11). The functions $\Psi_i(x)=(\nabla\varphi_i)(x)$ for $1\leqslant i\leqslant N$ are in $\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ and satisfy $\Psi_i(x_j)=\delta_{i,j}Z_i$ and $rot(\Psi_i)=rot(\nabla\varphi_i)=0$ for $i,j=1,\ldots,N$. It follows that the function $\Psi_Z(x)=\sum_{i=1}^{N}\Psi_i(x)$ is an element of $\mathcal{D}(\mathbb{R}^3,\mathbb{R}^3)$ satisfying $A\Psi_Z=Z$ and $rot(\Psi_Z)=0$. For all $Z=(Z_i)_{1\leqslant i\leqslant N}$ with $Z_i=(z_{i,k})_{1\leqslant k\leqslant 3}\in\mathbb{R}^3$ the functions $w_i(x)=(z_{i,2}t_3,z_{i,3}t_1,z_{i,1}t_2)$ satisfy $rot(w_i)=Z_i$. Let $\psi_i(x)=\phi_i(x)w_i(x)$ for $1\leqslant i\leqslant N$, where the functions ϕ_i are given by (2.11). The functions $\Phi_i(x)=rot(\psi_i)(x)$ are in $\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$. They satisfy $\Phi_i(x_j)=\delta_{i,j}Z_i$ and $div(\Phi_i)=div(rot(\psi_i))=0$ for $i,j=1,\ldots,N$. Then, the function $\Phi_Z(x)=\sum_{i=1}^{N}\Phi_i(x)$ is an element of $\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ satisfying $A\Phi_Z=Z$ and $div(\Phi_Z)=0$.

(5) Is an immediate consequence of item 4 and the fact that $\mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ is a subspace of $\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$.

(6) Let (f_k) be a sequence in $T(\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3))$ which converges in $(\mathcal{Y},\|\,.\,\|_{\mathcal{Y}})$ to an element f belonging to $\mathcal{Y}$. There exists a sequence (u_k) in $\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$ such that $Tu_k=f_k$ and (Tu_k) is a Cauchy sequence in $(\mathcal{Y},\|\,.\,\|_{\mathcal{Y}})$. Relation (2.24) together with Propositions 2.4 and 2.2 show that there exists an element $u\in\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$ such that (Tu_k) converges to Tu in $(\mathcal{Y},\|\,.\,\|_{\mathcal{Y}})$. Thus $f=Tu$ and $T(\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3))$ is a closed subspace in $(\mathcal{Y},\|\,.\,\|_{\mathcal{Y}})$.

(7) The subspace $T(\mathcal{V}^m(\mathbb{R}^3))$ is closed in $\mathcal{Y}$ and $Ker(T)=\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$ is a finite-dimensional space. Then for every closed subspace $\mathcal{W}$ of $\mathcal{V}^m(\mathbb{R}^3)$, the subspace $T(\mathcal{W})$ is closed in $\mathcal{Y}$ if and only if $\mathcal{W}+Ker(T)$ is closed (see [11]).

3. Vectorial approximation problem

First, we give the following.

Definition 3.1. For all $Z\in\mathbb{R}^{3N}$, $\rho>0$, $\tau>0$ and $\varepsilon\geqslant 0$ we define a vectorial tension spline function (VT-spline) as a solution $\sigma^{\varepsilon,\tau,\rho}$ of the following approximation problem:

$$\mathcal{P}_\varepsilon(Z):\quad \min_{v\in\mathcal{I}^m_\varepsilon(Z)}\Big(J_{m,\tau,\rho}(v)+\varepsilon\|Av-Z\|^2_{\mathbb{R}^{3N}}\Big), \tag{3.1}$$

where

$$\mathcal{I}_\varepsilon^m(Z) = \begin{cases} A^{-1}(Z) & \text{for } \varepsilon = 0 \text{ (interpolating problem)}, \\ \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3) & \text{for } \varepsilon > 0 \text{ (smoothing problem)} \end{cases} \tag{3.2}$$

$A^{-1}(Z) = \{v \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3) : A\,v = Z\}$ and $||\,.\,||_{\mathbb{R}^{3N}}$ is the Euclidean norm in $\mathbb{R}^{3N}$.

If $\rho = 1$, Proposition 2.4 shows that problem (3.1) splits into the following three separate problems, for $i = 1, 2, 3$:

$$\min_{f \in I_\varepsilon^m(Z_i)} \Big(|f|^2_{X^m} + \varepsilon ||af - Z_i||^2_{\mathbb{R}^N} \Big), \tag{3.3}$$

where

$$I_\varepsilon^m(Z_i) = \begin{cases} a^{-1}(Z_i) \text{ for } \varepsilon = 0 \text{ (interpolating problem)}, \\ X^m(\mathbb{R}^3) \text{ for } \varepsilon > 0 \text{ (smoothing problem)} \end{cases}$$

and $Z = (Z_1, Z_2, Z_3)$ with $Z_i = (z_{i,1}, \ldots, z_{i,n}) \in \mathbb{R}^{3N}$ for $i = 1, 2, 3$. The solutions of the three problems (3.3) are the spline under tension given in [5]. Hereafter, we suppose that $\rho \neq 1$; then problem (3.1) is coupled on the three components.

The following theorem gives the existence and uniqueness together with the characterization of the *VT*-spline.

Theorem 3.1. *For all* $Z \in \mathbb{R}^{3N}$, $\rho > 0$, $\tau > 0$ *and* $\varepsilon \geqslant 0$, *there is a unique* VT*-spline solution of problem* (3.1). *The VT-spline is the unique element* $\sigma^{\varepsilon,\tau,\rho}$ *of* $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ *characterized by the relation*

$$\langle T\sigma^{\varepsilon,\tau,\rho} | Tu \rangle_{\mathcal{Y}} + \varepsilon \langle A\sigma^{\varepsilon,\tau,\rho} - Z | Au \rangle_{\mathbb{R}^{3N}} = 0 \quad \forall u \in \mathcal{I}_\varepsilon^m(0), \tag{3.4}$$

where $\mathcal{I}_\varepsilon^m(0)$ *is given by* (3.2).

Proof. The existence, uniqueness and characterization of the solution $\sigma^{\varepsilon,\tau,\rho}$ of the problem (3.1) are immediate consequences of Proposition 2.5 and the general spline theory (see [4,11]). □

In order to give an explicit expression of the solution of problem (3.1), we introduce the following differential matrix-operators $P_{m,\tau,\rho}(D)$ and $Q_\rho(D)$ given by $\forall u = (u_1, u_2, u_3) \in \mathcal{D}'(\mathbb{R}^3; \mathbb{R}^3)$:

$$\begin{aligned} P_{m,\tau,\rho}(D)\,u &= \Delta_{m-1,\tau}\Big[-\Delta u + (1-\rho)\nabla(\mathit{div}\,u)\Big], \\ Q_\rho(D)\,u &= -\Delta u + \left(1 - \frac{1}{\rho}\right)\nabla(\mathit{div}\,u). \end{aligned} \tag{3.5}$$

The differential operators $P_{m,\tau,\rho}(D)$ and $Q_\rho(D)$ are the differential matrix-operators with components given by

$$P^{(i,j)}_{m,\tau,\rho}(D) = \Delta_{m-1,\tau}\Big[-\delta_{i,j}\Delta + (1-\rho)\partial^2_{i,j}\Big]$$

and

$$Q_\rho^{(i,j)}(D) = -\delta_{i,j}\Delta + \left(1 - \frac{1}{\rho}\right)\partial_{i,j}^2$$

for $i,\ j = 1, 2, 3$. They satisfy the following relation:

$$P_{m,\tau,\rho}(D)\,.\,Q_\rho(D) = Q_\rho(D)\,.\,P_{m,\tau,\rho}(D) = \Delta_{m+1,\tau} I_3, \tag{3.6}$$

where I_3 is the 3-unit matrix and $\Delta_{m+1,\tau}$ is given by (2.3).

Proposition 3.1. *For all* $u \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ *we have*:

(1) $J_{m,\tau,\rho}(u, \varphi) = \langle P_{m,\tau,\rho}(D)\, u, \varphi\rangle = \langle u, and\, P_{m,\tau,\rho}(D)\, \varphi\rangle$, *and*
(2) $P_{m,\tau,\rho}(D)\, u = 0$ *if and only if* $u \in \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$.

Proof. (1) From (3.5), we can obviously verify that, for $u = (u_1, u_2, u_3) \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ and $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$, we have

$$\begin{aligned} D_{m,\tau}(u, \varphi) &= -\sum_{i,j=1}^{3} \langle \Delta_{m-1,\tau}(\partial_{ij}^2) u_j, \varphi_i\rangle = -\langle \Delta_{m-1,\tau}[\nabla(div\, u)], \varphi\rangle \\ &= -\sum_{i,j=1}^{3} \langle u_i, \Delta_{m-1,\tau}(\partial_{ij}^2)\varphi_j\rangle = -\langle u, \Delta_{m-1,\tau}[\nabla(div\, \varphi)]\rangle \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} R_{m,\tau}(u, \varphi) &= \sum_{i,j=1}^{3} \langle \Delta_{m-1,\tau}(-\delta_{ij}\Delta + \partial_{ij}^2) u_j, \varphi_i\rangle = \langle \Delta_{m-1,\tau}[rot(rot\, u)], \varphi\rangle \\ &= \sum_{i,j=1}^{3} \langle u_i, \Delta_{m-1,\tau}(-\delta_{ij}\Delta + \partial_{ij}^2)\varphi_j\rangle = \langle u, \Delta_{m-1,\tau}[rot(rot\, \varphi)]\rangle. \end{aligned} \tag{3.8}$$

The goal is now achieved by taking into account the formula $rot(rot\, u) = \nabla(div\, u) - \Delta u$.

(2) It is clear that $P_{m,\tau,\rho}(D)\, u = 0$ for $u \in \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$. Conversely, let $u \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ such that $P_{m,\tau,\rho}(D)\, u = 0$. Then $\langle P_{m,\tau,\rho}(D).u, \varphi\rangle = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ and from item (1) we obtain $\rho D_{m,\tau}(u, \varphi) + R_{m,\tau}(u, \varphi) = 0$. So for all $v \in \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3) + \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ we also have $\rho D_{m,\tau}(u, v) + R_{m,\tau}(u, v) = 0$. The density of $\mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3) + \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ in $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ implies that for all $v \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ we have $\rho D_{m,\tau}(u, v) + R_{m,\tau}(u, v) = 0$. In particular, for $v = u$ and with relation (2.24), we obtain that $u \in Ker(T) = \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$. □

Let $\mu = (\mu_1, \mu_2, \mu_3)$ be a vectorial compactly supported measure. Then we say that μ is orthogonal to $\mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ if for all $p = (p_1, p_2, p_3) \in \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$, we have

$$\langle \mu, p\rangle = \sum_{i=1}^{3} \langle \mu_i, p_i\rangle = \sum_{i=1}^{3} \int_{\mathbb{R}^3} p_i(x)\, d\mu_i(x) = 0.$$

The next proposition gives some characterizations of the *VT*-spline.

Proposition 3.2. *Let* $\sigma^{\varepsilon,\tau,\rho}$ *be the* VT*-spline solution of problem* (3.1); *then there are unique coefficients* $\lambda_{i,j}^{\varepsilon,\tau,\rho}$ *with* $i = 1, 2, 3$ *and* $j = 1, \ldots, N$ *such that*

$$\mu^{\varepsilon,\tau,\rho} = \left(\sum_{j=1}^{N} \lambda_{1,j}^{\varepsilon,\tau,\rho} \delta_{x_j}, \sum_{j=1}^{N} \lambda_{2,j}^{\varepsilon,\tau,\rho} \delta_{x_j}, \sum_{j=1}^{N} \lambda_{3,j}^{\varepsilon,\tau,\rho} \delta_{x_j} \right) \tag{3.9}$$

is a vectorial-measure orthogonal to the space $\mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ *and*

$$P_{m,\tau,\rho}(D)\, \sigma^{\varepsilon,\tau,\rho} = \mu^{\varepsilon,\tau,\rho}, \tag{3.10}$$

where δ_{x_j} *denotes Dirac's measure at the point* x_j.

Proof. For $\varepsilon > 0$ (the smoothing problem case), we have $\mathcal{I}_\varepsilon^m(0) = \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$. By taking $u = \varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ in relation (3.4) we obtain

$$\langle T\sigma^{\varepsilon,\tau,\rho} | T\varphi \rangle_{\mathcal{Y}} = -\varepsilon \langle A\sigma^{\varepsilon,\tau,\rho} - Z | A\varphi \rangle_{\mathbb{R}^{3N}} \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3).$$

From relation (2.24), we obtain

$$\langle T\sigma^{\varepsilon,\tau,\rho} | T\varphi \rangle_{\mathcal{Y}} = \rho D_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}, \varphi) + R_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}, \varphi) \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3).$$

According to Proposition 3.1, we obtain

$$\langle P_{m,\tau,\rho}(D)\, \sigma^{\varepsilon,\tau,\rho}, \varphi \rangle = -\varepsilon \langle A\sigma^{\varepsilon,\tau,\rho} - Z | A\varphi \rangle_{\mathbb{R}^{3N}} \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3),$$

namely for all $\varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ we have

$$\langle P_{m,\tau,\rho}(D)\, \sigma^{\varepsilon,\tau,\rho}, \varphi \rangle = \sum_{i=1}^{3} \sum_{j=1}^{N} \varepsilon \big(z_{i,j} - \sigma_i^{\varepsilon,\tau,\rho}(x_j) \big) \varphi_i(x_j) = \sum_{i=1}^{3} \sum_{j=1}^{N} \lambda_{i,j}^{\varepsilon,\tau,\rho} \varphi_i(x_j),$$

where $\lambda_{i,j}^{\varepsilon,\tau,\rho} = \varepsilon\big(z_{i,j} - \sigma_i^{\varepsilon,\tau,\rho}(x_j)\big)$ for $i = 1, 2, 3$, $j = 1, \ldots, N$ and $\sigma^{\varepsilon,\tau,\rho} = (\sigma_1^{\varepsilon,\tau,\rho}, \sigma_2^{\varepsilon,\tau,\rho}, \sigma_3^{\varepsilon,\tau,\rho})$. The last relation may be written in the following form:

$$\langle P_{m,\tau,\rho}(D)\, \sigma^{\varepsilon,\tau,\rho}, \varphi \rangle = \sum_{i=1}^{3} \left\langle \sum_{j=1}^{N} \lambda_{i,j}^{\varepsilon,\tau,\rho} \delta_{x_j}, \varphi_i \right\rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3). \tag{3.11}$$

Now for $\varepsilon = 0$ (the interpolating problem case), we have $\mathcal{I}_0^m(0) = A^{-1}(0) = Ker(A)$. Let $v = (v_1, v_2, v_3)$ be any element of $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$, and consider the element $u = (u_1, u_2, u_3) = v - \psi \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ where $\psi = (\psi_1, \psi_2, \psi_3) \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$ is given by $\psi_i = \sum_{j=1}^{N} v_i(x_j)\phi_j$ for $i = 1, 2, 3$ and the functions $\phi_1, \ldots, \phi_N$ are given by (2.11). It can obviously be verified that u belongs to $Ker(A)$. From relation (3.4), we obtain that

$\langle T\sigma^{0,\tau,\rho}|Tu\rangle_{\mathcal{Y}} = 0$. By virtue of relation (2.24) together with Proposition 3.1, we obtain

$$\langle T\sigma^{0,\tau,\rho}|Tv\rangle_{\mathcal{Y}} = \langle T\sigma^{0,\tau,\rho}|T\psi\rangle_{\mathcal{Y}}$$
$$= \langle P_{m,\tau,\rho}(D).\sigma^{0,\tau,\rho}, \psi\rangle = \sum_{i=1}^{3} \langle (P_{m,\tau,\rho}(D).\sigma^{0,\tau,\rho})_i, \psi_i\rangle.$$

Now according to the representation of the components of ψ, we obtain that

$$\langle T\sigma^{0,\tau,\rho}|Tv\rangle_{\mathcal{Y}} = \sum_{i=1}^{3}\sum_{j=1}^{N} \langle (P_{m,\tau,\rho}(D).\sigma^{0,\tau,\rho})_i, \phi_j\rangle v_i(x_j),$$

which may be written in the following form:

$$\langle T\sigma^{0,\tau,\rho}|Tv\rangle_{\mathcal{Y}} = \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda_{i,j}^{0,\tau,\rho} v_i(x_j) \quad \forall v \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3), \tag{3.12}$$

where $\lambda_{i,j}^{0,\tau,\rho} = \langle (P_{m,\tau,\rho}(D)\,\sigma^{0,\tau,\rho})_i, \phi_j\rangle$ for $i = 1, 2, 3$ and $j = 1, \ldots, N$. In particular, for all $\varphi = (\varphi_1, \varphi_2, \varphi_3) \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3)$, we also have

$$\langle T\sigma^{0,\tau,\rho}|T\varphi\rangle_{\mathcal{Y}} = \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda_{i,j}^{0}\varphi_i(x_j) = \sum_{i=1}^{3} \left\langle \sum_{j=1}^{N} \lambda_{i,j}^{0,\tau,\rho}\delta_{x_j}, \varphi_i \right\rangle. \tag{3.13}$$

Again, by using relation (2.24) and Proposition 3.1, we obtain that

$$\langle P_{m,\tau,\rho}(D)\,\sigma^{0,\tau,\rho}, \varphi\rangle = \sum_{i=1}^{3} \left\langle \sum_{j=1}^{N} \lambda_{i,j}^{0,\tau,\rho}\delta_{x_j}, \varphi_i \right\rangle \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^3; \mathbb{R}^3). \tag{3.14}$$

Relations (3.11) and (3.14) prove the required result (3.10).

Let $p = (p_1, p_2, p_3) \in \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$; for $\varepsilon > 0$, by relation (3.4), we obtain

$$\sum_{i=1}^{3}\sum_{j=1}^{N} \lambda_{i,j}^{\varepsilon,\tau,\rho} p_i(x_j) = -\varepsilon\langle A\sigma^{\varepsilon,\tau,\rho} - Z|Ap\rangle_{\mathbb{R}^{3N}} = \langle T\sigma^{\varepsilon,\tau,\rho}|Tp\rangle_{\mathcal{Y}} = 0.$$

For $\varepsilon = 0$, by using relation (3.12), we obtain $\sum_{i=1}^{3}\sum_{j=1}^{N} \lambda_{i,j}^{0,\tau,\rho} p_i(x_j) = \langle T\sigma^{0,\tau,\rho}|Tp\rangle_{\mathcal{Y}}$ $= 0$, which proves the orthogonality condition

$$\langle \mu^{\varepsilon,\tau,\rho}, p\rangle = \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda_{i,j}^{\varepsilon,\tau,\rho}\langle\delta_{x_j}, p\rangle$$
$$= \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda_{i,j}^{\varepsilon,\tau,\rho} p(x_j) = 0 \quad \forall p \in \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3). \qquad \square$$

Let $F_{m,\tau,\rho}$ be a 3×3 matrix valued function given by

$$\begin{aligned} F_{m,\tau,\rho} = Q_\rho(D)\, E_{m+1} &= (Q_\rho^{(i,j)} E_{m+1})_{\substack{1 \leqslant i \leqslant 3 \\ j \leqslant j \leqslant 3}} \\ &= \left(-\delta_{i,j} \Delta\, E_{m+1} + \left(1 - \frac{1}{\rho}\right) \partial^2_{i,j} E_{m+1}\right)_{\substack{1 \leqslant i \leqslant 3 \\ j \leqslant j \leqslant 3}}, \end{aligned} \tag{3.15}$$

where E_{m+1} is the fundamental solution of the operator $\Delta_{m+1,\tau}$ given by relation (2.3). We have

$$\begin{aligned} P_{m,\tau,\rho}(D)\,.\,F_{m,\tau,\rho} &= (P_{m,\tau,\rho}(D)\,.\,Q_\rho(D))E_{m+1} \\ &= (\Delta_{m+1,\tau} I_3)E_{m+1} = (\Delta_{m+1,\tau} E_{m+1})I_3 = \delta I_3. \end{aligned} \tag{3.16}$$

Relation (3.16) means that the matrix-valued function $F_{m,\tau,\rho}$ is a fundamental solution of the differential matrix-operator $P_{m,\tau,\rho}(D)$.

We have the following.

Proposition 3.3. *For all vectorial compactly supported measures $\mu = (\mu_1, \mu_2, \mu_3)$ orthogonal to the space $\mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$, the convolution product $F_{m,\tau,\rho} * \mu$ belongs to $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$.*

Proof. It is an immediate consequence of Proposition 2.1. □

The following theorem gives the expression of the solution of problem (3.1).

Theorem 3.2. *Let $\sigma^{\varepsilon,\tau,\rho}$ be the VT-spline solution of problem* (3.1). *Then there exists a polynomial q in $\mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ such that*

$$\sigma^{\varepsilon,\tau,\rho} = F_{m,\tau,\rho} * \mu^{\varepsilon,\tau,\rho} + q, \tag{3.17}$$

where $\mu^{\varepsilon,\tau,\rho}$ is the vector-measure given by (3.9) *and $F_{m,\tau,\rho}$ is the matrix-function given by* (3.15). *Namely, the VT-spline $\sigma^{\varepsilon,\tau,\rho} = (\sigma_1^{\varepsilon,\tau,\rho}, \sigma_2^{\varepsilon,\tau,\rho}, \sigma_3^{\varepsilon,\tau,\rho})$ is explicitly given by*

$$\begin{aligned} \sigma_k^{\varepsilon,\tau,\rho}(x) = &\sum_{i=1}^{3} \sum_{j=1}^{N} \lambda_{i,j}^{\varepsilon,\tau,\rho} \left[\delta_{i,k} \Delta E_{m+1}(x - x_j) + \left(\frac{1}{\rho} - 1\right) \partial^2_{i,k} E_{m+1}(x - x_j)\right] \\ &+ \sum_{j=1}^{d(m)} \alpha_{k,j}^{\varepsilon,\tau,\rho} q_j(x) \end{aligned}$$

for $k = 1, 2, 3$, where $(q_1, \ldots, q_{d(m)})$ is a basis of the space $\mathbb{P}_{m-1}(\mathbb{R})$. The coefficients $\lambda_{k,i}^{\varepsilon,\tau,\rho}$ and $\alpha_{k,j}^{\varepsilon,\tau,\rho}$ for $k = 1, 2, 3$, $i = 1, \ldots, N$ and $j = 1, \ldots, d(m)$ are computed by solving the $3(N + d(m)) \times 3(N + d(m))$ nonsingular linear system

$$\begin{pmatrix} K + c_\varepsilon I_{3N} & M \\ M^T & O \end{pmatrix} \begin{pmatrix} \Lambda^{\varepsilon,\tau,\rho} \\ \alpha^{\varepsilon,\tau,\rho} \end{pmatrix} = \begin{pmatrix} Z \\ O \end{pmatrix} \quad \text{with} \quad c_\varepsilon = \begin{cases} \dfrac{1}{\varepsilon} & \text{if } \varepsilon > 0, \\ 0 & \text{if } \varepsilon = 0, \end{cases} \tag{3.18}$$

where $\Lambda^{\varepsilon,\tau,\rho}$ and $\alpha^{\varepsilon,\tau,\rho}$ are the vectors given by

$$\Lambda^{\varepsilon,\tau,\rho} = (\lambda_{1,1}^{\varepsilon,\tau,\rho}, \ldots, \lambda_{1,N}^{\varepsilon,\tau,\rho}, \lambda_{2,1}^{\varepsilon,\tau,\rho}, \ldots, \lambda_{2,N}^{\varepsilon,\tau,\rho}, \lambda_{3,1}^{\varepsilon,\tau,\rho}, \ldots, \lambda_{3,N}^{\varepsilon,\tau,\rho})^T \in \mathbb{R}^{3N},$$
$$\alpha^{\varepsilon,\tau,\rho} = (\alpha_{1,1}^{\varepsilon,\tau,\rho}, \ldots, \alpha_{1,d(m)}^{\varepsilon,\tau,\rho}, \alpha_{2,1}^{\varepsilon,\tau,\rho}, \ldots, \alpha_{2,d(m)}^{\varepsilon,\tau,\rho}, \alpha_{3,1}^{\varepsilon,\tau,\rho}, \ldots, \alpha_{3,d(m)}^{\varepsilon,\tau,\rho})^T \in \mathbb{R}^{3d(m)},$$

and I_{3N}, $K = (K_{l,k})_{1\leqslant l,k\leqslant 3}$ and $M = (M_{l,k})_{1\leqslant l,k\leqslant 3}$ are the $3N$-unit matrix, $3N \times 3N$ and $3N \times 3d(m)$ matrices, respectively. The blocks $K_{l,k}$ and $M_{l,k}$ of K and M are given by

$$K_{l,k} = \left(\delta_{l,k}\Delta E_{m+1}(x_i - x_j) + \left(\frac{1}{\rho} - 1\right)\partial_{l,k}^2 E_{m+1}(x_i - x_j)\right)_{1\leqslant i,j\leqslant N},$$
$$M_{l,k} = \delta_{l,k}\left(q_j(x_i)\right)_{\substack{1\leqslant i\leqslant N\\ 1\leqslant j\leqslant d(m)}},$$

respectively.

Proof. From Proposition 3.2, $P_{m,\tau,\rho}(D).\sigma^{\varepsilon,\tau,\rho}$ is a vectorial compactly supported measure $\mu^{\varepsilon,\tau,\rho}$ orthogonal to $\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$. From Proposition 3.3, the convolution product $F_{m,\tau,\rho} * \mu^{\varepsilon,\tau,\rho}$ belongs to $\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$ and we have

$$\begin{aligned} P_{m,\tau,\rho}(D)(F_{m,\tau,\rho} * \mu^{\varepsilon,\tau,\rho}) &= P_{m,\tau,\rho}(D)(F_{m,\tau,\rho}) * \mu^{\varepsilon,\tau,\rho} \\ &= \delta I_3 * \mu^{\varepsilon,\tau,\rho} = \mu^{\varepsilon,\tau,\rho} = P_{m,\tau,\rho}(D)(\sigma^{\varepsilon,\tau,\rho}), \end{aligned}$$

which gives

$$P_{m,\tau,\rho}(D)(\sigma^{\varepsilon,\tau,\rho} - F_{m,\tau,\rho} * \mu^{\varepsilon,\tau,\rho}) = 0.$$

According to Proposition 3.1, we obtain that $\sigma^{\varepsilon,\tau,\rho} - F_{m,\tau,\rho} * \mu^{\varepsilon,\tau,\rho}$ belongs to $\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3)$. This gives representation (3.17) of $\sigma^{\varepsilon,\tau,\rho}$ and with respect to the interpolating (and smoothing) conditions together with the orthogonality conditions, we obtain the linear system (3.18). □

4. Particular approximation problems

In this section we study four particular vectorial approximation problems. The first and second problems are the limit problems (as will be shown) when $\rho \to 0$ and $\rho \to +\infty$, respectively. The third and fourth are the divergence-free problem and the rotation-free problem, respectively. Problems similar to the third and fourth problems were studied by Dodu [8] and Handscomb [10]. In order to study the four cases simultaneously, we introduce a subscript ℓ which is, in the context of this section, equal to 1, 2, 3 or 4. We define the following vector subspaces of $\mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$ by

$$\begin{aligned} \mathcal{V}_1^m(\mathbb{R}^3;\mathbb{R}^3) &= \{v \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3),\ R_{m,\tau}(v) = 0\}, \\ \mathcal{V}_2^m(\mathbb{R}^3;\mathbb{R}^3) &= \{v \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3),\ D_{m,\tau}(v) = 0\}, \\ \mathcal{V}_3^m(\mathbb{R}^3;\mathbb{R}^3) &= \{v \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3),\ \mathit{rot}\, v = 0\}, \\ \mathcal{V}_4^m(\mathbb{R}^3;\mathbb{R}^3) &= \{v \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3),\ \mathit{div}\, v = 0\}, \end{aligned} \tag{4.1}$$

and we consider the quadratic form $J^{(\ell)}_{m,\tau}(v) = J^{(\ell)}_{m,\tau}(v, v)$ where

$$J^{(\ell+2)}_{m,\tau}(u, v) = J^{(\ell)}_{m,\tau}(u, v) = \delta_{1,\ell} D_{m,\tau}(u, v) + \delta_{2,\ell} R_{m,\tau}(u, v) \quad \text{for } \ell = 1, 2.$$

Since the linear operators *rot*, *div* and the quadratic form $J^{(\ell)}_{m,\tau}$ are continuous, then $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$ are closed subspaces of $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ and $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$ possessing the scalar product induced by the scalar product $\langle\, .|.\,\rangle_{\mathcal{V}^m}$ given by (2.15) are Hilbert spaces for $\ell = 1, 2, 3$ and 4.

It can obviously be verified that $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3) = \mathcal{V}^m_3(\mathbb{R}^3; \mathbb{R}^3) + \mathcal{V}^m_4(\mathbb{R}^3; \mathbb{R}^3)$, which leads to the fact that each field $v \in \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ may be decomposed into the sum

$$v = \nabla\chi + rot\,\psi, \tag{4.2}$$

with a rotation-free part $\nabla\chi$ (we have $rot\,\nabla\chi = 0$) and a divergence-free part $rot\,\psi$ (we have $div(rot\,\psi) = 0$). In fluid mechanics, Eq. (4.2) is known as the Helmholtz decomposition. The function χ is called the velocity potential and ψ is called the stream function.

4.1. Limit problems, divergence-free problem and rotation-free problem

Definition 4.1. For all $Z \in \mathbb{R}^{3N}$, $\tau > 0$, $\varepsilon \geqslant 0$ and $\ell = 1, 2, 3, 4$, we define the particular vectorial tension spline $\sigma^{\varepsilon,\tau}_\ell$ as a solution of the following approximation problem:

$$\mathcal{P}^{(\ell)}_\varepsilon(Z): \quad \min_{v\in\mathcal{I}^m_{\varepsilon,\ell}(Z)} (J^{(\ell)}_{m,\tau}(v) + \varepsilon \parallel Av - Z \parallel^2_{\mathbb{R}^{3N}}), \tag{4.3}$$

where

$$\mathcal{I}^m_{\varepsilon,\ell}(Z) = \begin{cases} A^{-1}(Z) \cap \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3) & \text{for } \varepsilon = 0 \text{ (interpolating problem)},\\ \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3) & \text{for } \varepsilon > 0 \text{ (smoothing problem)}. \end{cases} \tag{4.4}$$

Let us first remark that problem (4.3) is equivalent to the problem

$$\min_{\mathcal{I}^m_{\varepsilon,\ell}(Z)} (D_{m,\tau}(v) + R_{m,\tau}(v) + \varepsilon \parallel Av - Z \parallel^2_{\mathbb{R}^{3N}}). \tag{4.5}$$

Let $\mathcal{Y}_\ell$ be the space $\mathcal{Y}_\ell = \mathcal{Y}_D$ if $\ell = 1, 3$ or $\mathcal{Y}_\ell = \mathcal{Y}_R$ if $\ell = 2, 4$, where $\mathcal{Y}_R$ and $\mathcal{Y}_D$ are given by (2.17). Let $T_\ell : \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3) \to \mathcal{Y}_\ell$ be the restriction to $(\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3), \langle\, .|.\,\rangle_{\mathcal{V}^m})$ of the linear operator T_D if $\ell = 1, 3$ or of T_R if $\ell = 2, 4$, where T_D and T_R are given by (2.22)–(2.23) and let $A_\ell : \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3) \to \mathbb{R}^{3N}$ be the restriction to $(\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3), \langle\, .|.\,\rangle_{\mathcal{V}^m})$ of the operator A. We recall the following relation given in (2.24):

$$\langle T_\ell u | T_\ell v\rangle_{\mathcal{Y}_\ell} = J^{(\ell)}_{m,\tau}(u, v), \quad \forall u, v \in \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3).$$

Let us denote by $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)$ for $\ell = 1, 2, 3, 4$, the following polynomial subspaces of $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$ given by

$$\mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3) = \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3) \cap \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3). \tag{4.6}$$

Proposition 4.1. *We have*

$$\begin{aligned}
\mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3) &= \mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3) \quad \text{for} \quad \ell=1,2,\\
\mathbb{P}_{m-1,3}(\mathbb{R}^3;\mathbb{R}^3) &= \{p\in\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3) | rot\, p=0\}\\
&= \{p\in\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3) | \exists q\in\mathbb{P}_m(\mathbb{R}^3): p=\nabla q\},\\
\mathbb{P}_{m-1,4}(\mathbb{R}^3;\mathbb{R}^3) &= \{p\in\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3) | div\, p=0\}\\
&= \{p\in\mathbb{P}_{m-1}(\mathbb{R}^3;\mathbb{R}^3) | \exists q\in\mathbb{P}_m(\mathbb{R}^3;\mathbb{R}^3): p=rot\, q\}.
\end{aligned}$$

The dimension $d_\ell(m)$ of the space $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3)$ is given by

$$d_\ell(m)=\begin{cases} 3d(m)=\dfrac{m(m+1)(m+2)}{2} & \text{if } \ell=1,2,\\ \dfrac{(m+1)(m+2)(m+3)}{6}-1 & \text{if } \ell=3,\\ \dfrac{m(m+1)(2m+7)}{6} & \text{if } \ell=4.\end{cases}$$

Proof. See [10,8]. □

We state the existence, uniqueness and characterization of the solution of problem (4.3).

Proposition 4.2. *For all $Z\in\mathbb{R}^{3N}$, $\tau>0$, $\varepsilon\geqslant 0$ and $\ell=1,2,3,4$, problem* (4.3) *has a unique solution. The solution of problem* (4.3) *is a unique element $\sigma_\ell^{\varepsilon,\tau}$ of $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$ characterized by the relation*

$$\langle T_\ell\sigma_\ell^{\varepsilon,\tau}|T_\ell u\rangle_{\mathcal{Y}_\ell}+\varepsilon\langle A_\ell\sigma_\ell^{\varepsilon,\tau}-Z|A_\ell u\rangle_{\mathbb{R}^{3N}}=0,\quad \forall u\in\mathcal{I}^m_{\varepsilon,\ell}(0), \tag{4.7}$$

where $\mathcal{I}^m_{\varepsilon,\ell}(0)$ is given by (4.4).

Proof. From Proposition 2.5, the operators T_ℓ and A_ℓ are continuous from $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$ to $\mathcal{Y}_\ell$ and to $\mathbb{R}^{3N}$, respectively. They satisfy the following:

(1) A_ℓ is surjective (Proposition 2.5, item 4),
(2) $T_\ell(\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3))$ is closed in $\mathcal{Y}_\ell$ (Proposition 2.5, item 7),
(3) $Ker\, T_\ell=\mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3)$ and $Ker\, A_\ell+Ker T_\ell$ is closed in $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$,
(4) $Ker A_\ell\cap Ker T_\ell=\{0\}$.

Then the required result is, as for Theorem 3.1, a consequence of spline theory. □

In order to give an explicit expression of the solution of problem (4.3), we introduce the following differential operator $P_{m,\tau}^{(\ell)}(D)$ given by

$$P_{m,\tau}^{(\ell)}(D)\,u=\begin{cases} -\Delta_{m,\tau}(div\, u) & \text{for } \ell=1,3,\\ \Delta_{m,\tau}(rot\, u) & \text{for } \ell=2,4.\end{cases} \tag{4.8}$$

Proposition 4.3. *For all* $u \in \mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$, *for all* $\varphi \in \mathcal{D}(\mathbb{R}^3)$ *if* $\ell = 1, 3$ *and for all* $\varphi \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$ *if* $\ell = 2, 4$, *we have*

(1) $$\langle P^{(\ell)}_{m,\tau}(D)\,u,\varphi\rangle = \begin{cases} J^{(\ell)}_{m,\tau}(u,\nabla\varphi) & \text{for } \ell = 1,3,\\ J^{(\ell)}_{m,\tau}(u, rot\,\varphi) & \text{for } \ell = 2,4.\end{cases}$$

(2) $P^{(\ell)}_{m,\tau}(D)\,u = 0$ *if and only if* $u \in \mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3)$.

Proof. (1) For $\ell = 1, 3$. From (3.7), for $u = (u_1, u_2, u_3) \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$ and $\varphi \in \mathcal{D}(\mathbb{R}^3)$, we have

$$\begin{aligned} J^{(\ell)}_{m,\tau}(u,\nabla\varphi) &= D_{m,\tau}(u,\nabla\varphi) = -\langle\Delta_{m-1,\tau}[\nabla(div\,u)],\nabla\varphi\rangle\\ &= \langle\Delta_{m-1,\tau}[\Delta(div\,u)],\varphi\rangle\\ &= -\langle\Delta_{m,\tau}(div\,u),\varphi\rangle. \end{aligned}$$

For $\ell = 2, 4$, from (3.8), for $u = (u_1, u_2, u_3) \in \mathcal{V}^m(\mathbb{R}^3;\mathbb{R}^3)$ and $\varphi = (\varphi_1,\varphi_2,\varphi_3) \in \mathcal{D}(\mathbb{R}^3;\mathbb{R}^3)$, we have

$$J^{(\ell)}_{m,\tau}(u, rot\,\varphi) = R_{m,\tau}(u,\nabla\varphi) = \langle\Delta_{m-1,\tau}[rot\,(rot\,u)], rot\,\varphi\rangle.$$

Since $rot(rot\,u) = \nabla(div\,u) - \Delta u$, we obtain $J^{(\ell)}_{m,\tau}(u, rot\,\varphi) = \langle\Delta_{m,\tau}(rot\,u),\varphi\rangle$.

(2) The result is obtained from Proposition 3.1 item (2) by using relations (2.24). □

Proposition 4.4. *Let* $\sigma^{\varepsilon,\tau}_\ell$ *be the vectorial spline solution of problem* (4.3) *with* $\ell = 1, 2, 3, 4$, *then there are unique coefficients* $\lambda^{\varepsilon,\tau}_{i,j,\ell}$ *with* $i = 1, 2, 3$ *and* $j = 1, \ldots, N$ *such that* $\mu^{\varepsilon,\tau}_\ell = \left(\sum_{j=1}^N \lambda^{\varepsilon,\tau}_{1,j,\ell}\delta_{x_j}, \sum_{j=1}^N \lambda^{\varepsilon,\tau}_{2,j,\ell}\delta_{x_j}, \sum_{j=1}^N \lambda^{\varepsilon,\tau}_{3,j,\ell}\delta_{x_j}\right)$ *is a vectorial measure orthogonal to the space* $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3)$ *and*

$$P^{(\ell)}_{m,\tau}(D)\,\sigma^{\varepsilon,\tau}_\ell = \begin{cases} div(\mu^{\varepsilon,\tau}_\ell) & \text{for } \ell = 1,3,\\ rot(\mu^{\varepsilon,\tau}_\ell) & \text{for } \ell = 2,4.\end{cases} \tag{4.9}$$

Proof. It is similar to the proof of Proposition 3.2 by using the characterization given by relation (4.7). □

The following theorem gives a characterization and computing method of the vectorial spline of the particular problems.

Proposition 4.5. *The unique solution* $\sigma^{\varepsilon,\tau}_\ell = (\sigma^{\varepsilon,\tau}_{1,\ell},\sigma^{\varepsilon,\tau}_{2,\ell},\sigma^{\varepsilon,\tau}_{3,\ell}) \in \mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$ *of problem* (4.5) *is explicitly given by*
for $k = 1, 2, 3$, *and* $\ell = 1, 3$,

$$\sigma^{\varepsilon,\tau}_{k,\ell}(x) = \sum_{i=1}^3\sum_{j=1}^N \lambda^{\varepsilon,\tau}_{i,j,\ell}\partial^2_{i,k}E_{m+1}(x - x_j) + \sum_{j=1}^{d_\ell(m)} \alpha^{\varepsilon,\tau}_{j,\ell}q_{k,j,\ell}(x),$$

and for $k = 1, 2, 3,$ *and* $\ell = 2, 4,$

$$\sigma^{\varepsilon,\tau}_{k,\ell}(x) = \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda^{\varepsilon,\tau}_{i,j,\ell}(-\delta_{i,k}\Delta + \partial^2_{i,k})E_{m+1}(x - x_j) + \sum_{j=1}^{d_\ell(m)} \alpha^{\varepsilon,\tau}_{j,\ell} q_{k,j,\ell}(x),$$

where $(q_{1,\ell}, \ldots, q_{d_\ell(m),\ell})$ *is a basis of* $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)$ *with* $q_{j,\ell} = (q_{1,j,\ell}, q_{2,j,\ell}, q_{3,j,\ell})$ *for* $j = 1, \ldots, d_\ell(m)$. *The coefficients* $\lambda^{\varepsilon,\tau}_{i,j,\ell}$ *and* $\alpha^{\varepsilon,\tau}_{k,\ell}$ for $i = 1, 2, 3,$ $j = 1, \ldots, N,$ $k = 1, \ldots, d_\ell(m)$ *and* $\ell = 1, 2, 3, 4$ *are computed by solving the* $(3N + d_\ell(m)) \times (3N + d_\ell(m))$ *linear system*

$$\begin{pmatrix} K^{(\ell)} + c_\varepsilon I_{3N} & M^{(\ell)} \\ M^{(\ell)T} & 0 \end{pmatrix} \begin{pmatrix} \Lambda^{\varepsilon,\tau}_\ell \\ \alpha^{\varepsilon,\tau}_\ell \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix} \quad \textit{with} \quad c_\varepsilon = \begin{cases} \dfrac{1}{\varepsilon} & \textit{if } \varepsilon > 0, \\ 0 & \textit{if } \varepsilon = 0, \end{cases}$$

where $\Lambda^{\varepsilon,\tau}_\ell$ *and* $\alpha^{\varepsilon,\tau}_\ell$ *are the vectors given by*

$$\Lambda^{\varepsilon,\tau}_\ell = (\lambda^{\varepsilon,\tau}_{1,1,\ell}, \ldots, \lambda^{\varepsilon,\tau}_{1,N,\ell}, \lambda^{\varepsilon,\tau}_{2,1,\ell}, \ldots, \lambda^{\varepsilon,\tau}_{2,N,\ell}, \lambda^{\varepsilon,\tau}_{3,1,\ell}, \ldots, \lambda^{\varepsilon,\tau}_{3,N,\ell})^T \in \mathbb{R}^{3N},$$
$$\alpha^{\varepsilon,\tau}_\ell = (\alpha^{\varepsilon,\tau}_{1,\ell}, \ldots, \alpha^{\varepsilon,\tau}_{d_\ell(m),\ell})^T \in \mathbb{R}^{d_\ell(m)}$$

and I_{3N} *is the* $3N$*-unit matrix,* $K^{(\ell)} = (K^{(\ell)}_{k,p})_{1 \leqslant k,p \leqslant 3}$ *is the* $3N \times 3N$ *matrix given by blocks*

$$K^{(\ell)}_{k,p} = \begin{cases} \left(\partial^2_{k,p} E_{m+1}(x_i - x_j)\right)_{1 \leqslant i,j \leqslant N} & \textit{for } \ell = 1, 3, \\ \left((-\delta_{k,p}\Delta + \partial_{k,p}) E_{m+1}(x_i - x_j)\right)_{1 \leqslant i,j \leqslant N} & \textit{for } \ell = 2, 4 \end{cases}$$

and $M^{(\ell)} = (M^{(\ell)}_1\ M^{(\ell)}_2\ M^{(\ell)}_3)^T$ *is the* $3N \times d_\ell(m)$ *matrix given by blocks*

$$M^{(\ell)}_k = (q_{k,j,\ell}(x_i))_{\substack{1 \leqslant i \leqslant N \\ 1 \leqslant j \leqslant d_\ell(m)}} \quad \textit{for } k = 1, 2, 3.$$

Proof. Let $\ell = 1$ or 3. Let $v = (v_1, v_2, v_3) := \nabla E_{m+1} * \mathit{div}(\mu^{\varepsilon,\tau}_\ell)$, where E_{m+1} is the fundamental solution given by (2.5) of the operator $\Delta_{m+1,\tau}$ and

$$v_p(x) = \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda^{\varepsilon,\tau}_{i,j,\ell} \partial^2_{i,p} E_{m+1}(x - x_j) \quad \text{for } p = 1, 2, 3.$$

Since the measure $\mu^{\varepsilon,\tau}_\ell$ is orthogonal to $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)$ and $\mathit{rot}\, v = 0$, it follows from Proposition 2.1, that the element v belongs to $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$. It can obviously be verified that $P^{(\ell)}_{m,\tau}(D)\, v = \mathit{div}\, \mu^{\varepsilon,\tau}_\ell = P^{(\ell)}_{m,\tau}(D)\, \sigma^{\varepsilon,\tau}_\ell$. This implies that $P^{(\ell)}_{m,\tau}(D)\, [\sigma^{\varepsilon,\tau}_\ell - v] = 0$. According to Proposition 4.3, we obtain that $\sigma^{\varepsilon,\tau}_\ell - v$ belongs to $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)$.

Let $\ell = 2$ or 4 and let $w = (w_1, w_2, w_3) := \mathit{rot}[\mathit{rot}((I_3 E_{m+1}) * \mu^{\varepsilon,\tau}_\ell)]$. From the relation $\mathit{rot}(\mathit{rot}\, u) = \nabla(\mathit{div}\, u) - \Delta u$, we have

$$w_p(x) = \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda^{\varepsilon,\tau}_{i,j,\ell}(-\delta_{i,p}\Delta + \partial^2_{i,p}) E_{m+1}(x - x_j), \quad \text{for } p = 1, 2, 3.$$

Since the measure $\mu_\ell^{\varepsilon,\tau}$ is orthogonal to $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3)$ and $div\, w = 0$, it follows from Proposition 2.1 that the element w belongs to $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$. It can obviously be verified that $P_{m,\tau}^{(\ell)}(D)\, w = rot\, \mu_\ell^{\varepsilon,\tau} = P_{m,\tau}^{(\ell)}(D)\, \sigma_\ell^{\varepsilon,\tau}$. This implies that $P_{m,\tau}^{(\ell)}(D)\,[\sigma_\ell^{\varepsilon,\tau} - w] = 0$. According to Proposition 4.3, we obtain that $\sigma_\ell^{\varepsilon,\tau} - w$ belongs to $\mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3)$.

The linear system is deduced in the same way as for Theorem 3.2. $\square$

4.2. Example: rotation-free approximation

In this subsection we give an explicit example for the rotation-free approximation problem which corresponds to the case $\ell = 3$ in Proposition 4.4. Let us set $m = 2$ and let $Z = (Z_1, Z_2, Z_3)$ be a given vector in $\mathbb{R}^{3N}$ where $Z_i = (z_{i,1}, \ldots, z_{i,N}) \in \mathbb{R}^N$ for $i = 1, 2, 3$. We consider

$$\mathcal{A} = \{x_i = (x_{1,i}, x_{2,i}, x_{3,i}) \in \mathbb{R}^3, \quad i = 1, \ldots, N\}$$

an ordered set of N distinct points of $\mathbb{R}^3$ with $N > d(2) = 4$. We suppose that the set $\mathcal{A}$ contains a $\mathbb{P}_1(\mathbb{R}^3)$-unisolvent subset of 4 points. The $\mathbb{P}_1(\mathbb{R}^3)$-unisolvence of $\mathcal{A}$ means that $\mathcal{A}$ contains at least 4 points which are not located in the same plane of $\mathbb{R}^3$. The case of the rotation-free approximation in problem (4.3) can be written as

$$\mathcal{P}_\varepsilon^{(3)}(Z): \qquad \min_{v=(v_1,v_2,v_3)\in\mathcal{I}^2_{\varepsilon,3}(Z)} (J^{(3)}_{2,\tau}(v) + \varepsilon \sum_{i=1}^{3}\sum_{j=1}^{N} |v_i(x_{i,j}) - z_{i,j}|^2),$$

where

$$\mathcal{I}^2_{0,3}(Z) = \{v = (v_1, v_2, v_3) \in \mathcal{V}_3^2(\mathbb{R}^3;\mathbb{R}^3) : v_i(x_{i,j}) = z_{i,j}, i = 1, 2, 3; \; j = 1, \ldots, N\}$$

for the interpolating problem ($\varepsilon = 0$) and

$$\mathcal{I}^2_{\varepsilon,3}(Z) = \mathcal{V}_3^2(\mathbb{R}^3;\mathbb{R}^3) := \{v \in \mathcal{V}^2(\mathbb{R}^3;\mathbb{R}^3) : rot\, v = 0\}$$

for the smoothing problem ($\varepsilon > 0$). The quadratic form $J^{(3)}_{2,\tau}$ is given by

$$J^{(3)}_{2,\tau}(v) = \sum_{|\alpha|=2} \frac{2!}{\alpha!} \int_{\mathbb{R}^3} |D^\alpha(div\, v)(x)|^2 dx + \tau^2 \sum_{|\alpha|=1} \int_{\mathbb{R}^3} |D^\alpha(div\, v)(x)|^2 dx.$$

Let us introduce the polynomials $q_{k,3}$ for $k = 1, \ldots, 9$ of the space $\mathbb{P}_1(\mathbb{R}^3;\mathbb{R}^3)$ given by

$$\begin{aligned} &q_{1,3}(x) = (1,0,0), \quad &&q_{2,3}(x) = (0,1,0), \quad &&q_{3,3}(x) = (0,0,1),\\ &q_{4,3}(x) = (x_1,0,0), &&q_{5,3}(x) = (0,x_2,0), &&q_{6,3}(x) = (0,0,x_3),\\ &q_{7,3}(x) = (0,x_3,x_2), &&q_{8,3}(x) = (x_3,0,x_1), &&q_{9,3}(x) = (x_2,x_1,0), \end{aligned}$$

for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. The family $(q_{1,3}, \ldots, q_{9,3})$ is a basis of the space

$$\mathbb{P}_{1,3}(\mathbb{R}^3;\mathbb{R}^3) = \{p \in \mathbb{P}_1(\mathbb{R}^3;\mathbb{R}^3) : rot\, p = 0\}$$

given in Proposition 4.1. The solution $\sigma_3^{\varepsilon,\tau} = (\sigma_{1,3}^{\varepsilon,\tau}, \sigma_{2,3}^{\varepsilon,\tau}, \sigma_{3,3}^{\varepsilon,\tau}) \in \mathcal{V}_3^2(\mathbb{R}^3; \mathbb{R}^3)$ of the problem $\mathcal{P}_\varepsilon^{(3)}(Z)$ is given by Proposition 4.5 as follows:

$$\sigma_3^{\varepsilon,\tau}(x) = \sum_{i=1}^{3}\sum_{j=1}^{N} \lambda_{i,j,3}^{\varepsilon,\tau} \partial_i (\nabla E_3)(x - x_j) + \sum_{k=1}^{9} \alpha_{k,3}^{\varepsilon,\tau} q_{k,3}(x),$$

where

$$E_3(x) = -\frac{1}{4\pi\tau^6 ||x||}\left(e^{-\tau||x||} - \sum_{k=0}^{4} \frac{(-\tau||x||)^k}{k!}\right)$$

is a fundamental solution of the operator $\Delta_{3,\tau}$. The coefficients $\lambda_{i,j,3}^{\varepsilon,\tau}$ and $\alpha_{k,3}^{\varepsilon,\tau}$ for $i = 1, 2, 3$; $j = 1, \dots, N$ and $k = 1, \dots, 9$ are computed by solving the $(3N + 9) \times (3N + 9)$ linear system

$$\begin{pmatrix} K^{(3)} + c_\varepsilon I_{3N} & M^{(3)} \\ M^{(3)T} & 0 \end{pmatrix} \begin{pmatrix} \Lambda_\ell^{\varepsilon,\tau} \\ \alpha_3^{\varepsilon,\tau} \end{pmatrix} = \begin{pmatrix} Z \\ 0 \end{pmatrix} \quad \text{with} \quad c_\varepsilon = \begin{cases} \dfrac{1}{\varepsilon} & \text{if } \varepsilon > 0, \\ 0 & \text{if } \varepsilon = 0. \end{cases}$$

The vectors $\Lambda_3^{\varepsilon,\tau}$ and $\alpha_3^{\varepsilon,\tau}$ are given by $\Lambda_3^{\varepsilon,\tau} = (\lambda_{1,1,3}^{\varepsilon,\tau}, \dots, \lambda_{1,N,3}^{\varepsilon,\tau}, \lambda_{2,1,3}^{\varepsilon,\tau}, \dots, \lambda_{2,N,3}^{\varepsilon,\tau}, \lambda_{3,1,3}^{\varepsilon,\tau}, \dots, \lambda_{3,N,3}^{\varepsilon,\tau})^T \in \mathbb{R}^{3N}$ and $\alpha_3^{\varepsilon,\tau} = (\alpha_{1,3}^{\varepsilon,\tau}, \dots, \alpha_{9,3}^{\varepsilon,\tau})^T \in \mathbb{R}^9$. The matrix I_{3N} is the $3N$-unit matrix and $K^{(3)} = (K_{k,p}^{(3)})_{1 \leqslant k,p \leqslant 3}$ is the $3N \times 3N$ block-matrix whose blocks are given by

$$K_{k,p}^{(3)} = \left(\partial_{k,p}^2 E_{m+1}(x_i - x_j)\right)_{1 \leqslant i,j \leqslant N}.$$

The matrix $M^{(3)} = \begin{pmatrix} M_1^{(3)} \\ M_2^{(3)} \\ M_3^{(3)} \end{pmatrix}$ is the $3N \times 9$ matrix given by

$$M_1^{(3)} = \begin{pmatrix} 1 & 0 & 0 & x_{1,1} & 0 & 0 & 0 & x_{3,1} & x_{2,1} \\ 1 & 0 & 0 & x_{1,2} & 0 & 0 & 0 & x_{3,2} & x_{2,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & x_{1,N} & 0 & 0 & 0 & x_{3,N} & x_{2,N} \end{pmatrix},$$

$$M_2^{(3)} = \begin{pmatrix} 0 & 1 & 0 & 0 & x_{2,1} & 0 & x_{3,1} & 0 & x_{1,1} \\ 0 & 1 & 0 & 0 & x_{2,2} & 0 & x_{3,2} & 0 & x_{1,2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 1 & 0 & 0 & x_{2,N} & 0 & x_{3,N} & 0 & x_{1,N} \end{pmatrix}$$

and

$$M_3^{(3)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & x_{3,1} & x_{2,1} & x_{1,1} & 0 \\ 0 & 0 & 1 & 0 & 0 & x_{3,2} & x_{2,2} & x_{1,2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 & 0 & x_{3,N} & x_{2,N} & x_{1,N} & 0 \end{pmatrix}.$$

4.3. Limit problems and convergence

In this subsection we study the convergence of problem (3.1) for $\rho \to 0$ and $\rho \to +\infty$. We will show that problem (3.1) converges for $\rho \to 0$ and $\rho \to +\infty$ to the limit problems $\mathcal{P}_\varepsilon^{(1)}(Z)$ and $\mathcal{P}_\varepsilon^{(2)}(Z)$ given by (4.3), respectively. In order to deal with this, we state the following general result.

Let $J : E \to \mathbb{R}$ be a functional defined on a topological space E, we recall that J is said to be lower semi-continuous ($l.s.c$) if and only if for all $\lambda \in \mathbb{R}$, the subset $S_\lambda = \{x \in E,\ J(x) \leqslant \lambda\}$ is closed in E.

Lemma 4.1. *Let $(E, ||\,.\,||_E)$ be a reflexive Banach space, let J_1 and J_2 be two positive convex and $l.s.c$ functionals on $(E, ||\,.\,||_E)$, let C be a convex closed nonempty subset of $(E, ||\,.\,||_E)$ and consider the following three minimum problems*:

$$\mathcal{P}_{\rho>0} : \inf_{v \in C} (J_1(v) + \rho J_2(v)),\ \mathcal{P}_\infty : \inf_{v \in C_1} J_1(v) \text{ and } \mathcal{P}_0 : \inf_{v \in C_2} J_2(v),$$

where $C_1 = \{v \in C \mid J_2(v) = 0\}$ and $C_2 = \{v \in C \mid J_1(v) = 0\}$ are nonempty. We suppose that

(1) *Each minimum problem $\mathcal{P}_{\rho>0}$, $\mathcal{P}_\infty$ and $\mathcal{P}_0$ admits a unique solution denoted by σ_ρ, σ_∞ and σ_0, respectively.*
(2) *The functional $J = J_1 + J_2$ is coercive, i.e., $\lim\limits_{\|v\|_E \to +\infty} J(v) = +\infty$.*

Then

(a) $\lim\limits_{\rho \to 0} J_1(\sigma_\rho) = J_1(\sigma_0) = 0$, $\lim\limits_{\rho \to 0} J_2(\sigma_\rho) = J_2(\sigma_0)$ *and* $\lim\limits_{\rho \to 0} \sigma_\rho = \sigma_0$ *weakly.*
(b) $\lim\limits_{\rho \to \infty} J_1(\sigma_\rho) = J_1(\sigma_\infty) = 0$, $\lim\limits_{\rho \to \infty} J_2(\sigma_\rho) = J_2(\sigma_\infty)$ *and* $\lim\limits_{\rho \to \infty} \sigma_\rho = \sigma_\infty$ *weakly.*

Proof. The proof is given for $\rho \to 0$. We have

$$J_1(\sigma_\rho) + \rho J_2(\sigma_\rho) \leqslant J_1(v) + \rho J_2(v) \quad \forall v \in C, \tag{4.10}$$

by taking $v = \sigma_0$ in equality (4.10) we obtain

$$J_1(\sigma_\rho) \leqslant \rho J_2(\sigma_0) \quad \text{and} \quad J_2(\sigma_\rho) \leqslant J_2(\sigma_0). \tag{4.11}$$

Then $\lim_{\rho \to 0} J_1(\sigma_\rho) = 0$ and consequently $(J_1(\sigma_\rho) + J_2(\sigma_\rho))_{\rho>0}$ is a bounded real sequence. Since the sum functional $J = J_1 + J_2$ is coercive, the sequence $(\sigma_\rho)_{\rho>0}$ is bounded in the space $(E, ||\,.\,||_E)$. Thus there is a subsequence $(\sigma_{\rho_n})_{n \in \mathbb{N}}$ of $(\sigma_\rho)_{\rho>0}$, which weakly converges to some limit $v_0 \in E$.

Since C is a convex closed subset, it follows that C is a weak closed set and v_0 belongs to C. The property of $l.s.c$ of J_1 implies that $J_1(v_0) \leqslant \liminf_{n\to+\infty} J_1(\sigma_{\rho_n}) = 0$, and consequently v_0 belongs to C_2. Also, the property of $l.s.c$ of J_2 together with inequality (4.11) implies that $J_2(v_0) \leqslant \liminf_{n\to+\infty} J_2(\sigma_{\rho_n}) \leqslant J_2(\sigma_0)$.

By uniqueness of the solution σ_0 of $\mathcal{P}_0$, we obtain that $v_0 = \sigma_0$. In the same manner, we can show that every weakly convergent subsequence of $(\sigma_\rho)_{\rho>0}$ weakly converges necessarily to σ_0 and consequently $\lim_{\rho\to 0} \sigma_\rho = \sigma_0$ weakly.

The problem $\mathcal{P}_\rho$ is equivalent to the problem $\inf_{v\in C} (\frac{1}{\rho} J_1(v) + J_2(v))$ and consequently the proof for $\rho \to +\infty$ is similar to the one for $\rho \to 0$. $\square$

Remark 4.1. The existence of the solution σ_ρ of the problem $\mathcal{P}_{\rho>0}$ is guaranteed by the fact that the functional J is coercive. In fact, in item (1) of Lemma 4.1, we can only suppose the uniqueness of the solution σ_ρ.

Now we are ready to state the following theorem.

Theorem 4.1. *The solution $\sigma^{\varepsilon,\rho,\tau}$ of problem* (3.1) *satisfies*

$$(1) \quad \lim_{\rho\to 0} ||\sigma^{\varepsilon,\tau,\rho} - \sigma_1^{0,\tau}||_{\mathcal{V}^m} = 0 \quad \textit{and} \quad (2) \quad \lim_{\rho\to\infty} ||\sigma^{\varepsilon,\tau,\rho} - \sigma_2^{\varepsilon,\tau}||_{\mathcal{V}^m} = 0,$$

where $\sigma_\ell^{\varepsilon,\tau}$ is the solution of problem (4.3) *for $\ell = 1, 2$, respectively.*

Proof. First, let us remark that, in the case $\varepsilon = 0$, the interpolating problem (3.1) is equivalent to the problem

$$\min_{\mathcal{I}_0^m(Z)} (\rho D_{m,\tau}(v) + R_{m,\tau}(v) + ||Av - Z||^2_{\mathbb{R}^{3N}}).$$

The proof of Theorem 4.1 is given by using Lemma 4.1. Let

$$J_1(v) = \begin{cases} R_{m,\tau}(v) + \varepsilon||Av - Z||^2_{\mathbb{R}^{3N}} & \text{if } \varepsilon > 0, \\ R_{m,\tau}(v) + ||Av - Z||^2_{\mathbb{R}^{3N}} & \text{if } \varepsilon = 0, \end{cases}$$

and $J_2(v) = D_{m,\tau}(v)$. J_1 and J_2 are two convex continuous functionals on $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$. We have $C = \mathcal{I}_\varepsilon^m(Z)$, $C_1 = \mathcal{I}_{\varepsilon,1}^m(Z)$ and $C_2 = \mathcal{I}_{\varepsilon,2}^m(Z)$. The minimum problems $\mathcal{P}_{\rho>0}$, $\mathcal{P}_\infty$ and $\mathcal{P}_0$ have, respectively, a unique solution $\sigma_\rho = \sigma^{\varepsilon,\tau,\rho}$, $\sigma_0 = \sigma_1^{0,\tau}$ and $\sigma_\infty = \sigma_2^{\varepsilon,\tau}$. The functional defined on $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ by $v \to (D_{m,\tau}(v) + R_{m,\tau}(v) + ||Av||^2_{\mathbb{R}^{3N}})^{1/2}$ is a norm in $\mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$ equivalent to the norm $||\,.\,||_{\mathcal{V}^m}$. Thus, $D_{m,\tau}(v) + R_{m,\tau}(v) + ||Av||^2_{\mathbb{R}^{3N}}$ converges to ∞ as $||v||_{\mathcal{V}^m} \to \infty$. Since

$$J_1(v) + J_2(v) \geqslant \min(1, \varepsilon)\Big(D_{m,\tau}(v) + R_{m,\tau}(v) + ||Av - Z||^2_{\mathbb{R}^{3N}}\Big)$$

and

$$||Av - Z||_{\mathbb{R}^{3N}} \geqslant \Big| ||Av||_{\mathbb{R}^{3N}} - ||Z||_{\mathbb{R}^{3N}} \Big|,$$

it follows that $J(v) = J_1(v) + J_2(v) \to \infty$ as $||v||_{\mathcal{V}^m} \to \infty$, which proves that the functional J is coercive. According to Lemma 4.1, we obtain

(a) $\lim_{\rho\to 0} J_1(\sigma^{\varepsilon,\tau,\rho}) = J_1(\sigma_1^{0,\tau}) = 0$, $\lim_{\rho\to 0} J_2(\sigma^{\varepsilon,\tau,\rho}) = J_2(\sigma_1^{0,\tau})$,

(b) $\lim_{\rho\to\infty} J_1(\sigma^{\varepsilon,\tau,\rho}) = J_1(\sigma_2^{\varepsilon,\tau}) = 0$, $\lim_{\rho\to\infty} J_2(\sigma^{\varepsilon,\tau,\rho}) = J_2(\sigma_2^{\varepsilon,\tau})$,

(c) $\lim_{\rho\to 0} \sigma^{\varepsilon,\tau,\rho} = \sigma_1^{0,\tau}$ and $\lim_{\rho\to\infty} \sigma^{\varepsilon,\tau,\rho} = \sigma_2^{\varepsilon,\tau}$ weakly.

Property (a) implies that

$$\lim_{\rho\to 0} R_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}) = 0 = R_{m,\tau}(\sigma_1^{0,\tau}), \quad \lim_{\rho\to 0} D_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}) = D_{m,\tau}(\sigma_1^{0,\tau}),$$

and $\lim_{\rho\to 0} A\sigma^{\varepsilon,\tau,\rho} = A\sigma_1^{0,\tau}$, in $\mathbb{R}^{3N}$. It follows that

$$\lim_{\rho\to 0}\Big(D_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}) + R_m(\sigma^{\varepsilon,\tau,\rho}) + ||A\sigma^{\varepsilon,\tau,\rho}||^2_{\mathbb{R}^{3N}}\Big) = D_{m,\tau}(\sigma_1^{0,\tau}) + R_{m,\tau}(\sigma_1^{0,\tau}) + \parallel A\sigma_1^{0,\tau} \parallel^2_{\mathbb{R}^{3N}},$$

and consequently, $\lim_{\rho\to 0} ||\sigma^{\varepsilon,\tau,\rho}||_{\mathcal{V}^m} = ||\sigma_1^{0,\tau}||_{\mathcal{V}^m}$. The weak convergence (property (c)) together with the norm convergence implies the strong convergence $\lim_{\rho\to 0} ||\sigma^{\varepsilon,\tau,\rho} - \sigma_1^{0,\tau}||_{\mathcal{V}^m} = 0$.

From property (c), we have the weak convergence $\lim_{\rho\to+\infty} \sigma^{\varepsilon,\tau,\rho} = \sigma_2^{\varepsilon,\tau}$ in $\mathcal{V}^m(\mathbb{R}^3)$. So, we obtain the weak and consequently the strong convergence $\lim_{\rho\to+\infty} A\sigma^{\varepsilon,\tau,\rho} = A\sigma_2^{\varepsilon,\tau}$ in $\mathbb{R}^{3N}$.

Property (b) implies that

$$\lim_{\rho\to\infty} D_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}) = D_{m,\tau}(\sigma_2^{\varepsilon,\tau}) \text{ and } \lim_{\rho\to\infty} R_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}) = R_{m,\tau}(\sigma_2^{\varepsilon,\tau}).$$

It follows that

$$\lim_{\rho\to\infty}\Big(D_{m,\tau}(\sigma^{\varepsilon,\tau,\rho}) + R_m(\sigma^{\varepsilon,\tau,\rho}) + ||A\sigma^{\varepsilon,\tau,\rho}||^2_{\mathbb{R}^{3N}}\Big) = D_{m,\tau}(\sigma_2^{\varepsilon,\tau}) + R_{m,\tau}(\sigma_2^{\varepsilon,\tau}) + \parallel A\sigma_2^{\varepsilon,\tau} \parallel^2_{\mathbb{R}^{3N}},$$

which concludes the proof. □

5. Convergence in Sobolev space

In order to avoid boundary conditions, as in the case of the scalar thin plate spline problem, variational problems (3.1) and (4.3) were set by considering a semi-norm involving integrals on the whole space $\mathbb{R}^3$. This allows an effective construction of the approximant. To show that the vectorial spline with tension may also be used for approximation in Sobolev space, on an open bounded set Ω of $\mathbb{R}^3$, we will study, in this section, the convergence problem.

Let Ω be a bounded open subset of $\mathbb{R}^3$ and consider the classical Sobolev space $H^{m+1}(\Omega)$ of order $m+1$. In $H^{m+1}(\Omega)$ we define the symmetric bilinear form

$$(f|g)_{m,\tau,\Omega} = \sum_{|\alpha|=m+1} \frac{(m+1)!}{\alpha!} \int_\Omega D^\alpha f(x) D^\alpha g(x) dx + \tau^2 \sum_{|\alpha|=m} \frac{m!}{\alpha!} \int_\Omega D^\alpha f(x) D^\alpha g(x) dx,$$

for $f, g \in H^{m+1}(\Omega)$. Consider the space $H^{m+1}(\Omega; \mathbb{R}^3) := H^{m+1}(\Omega) \times H^{m+1}(\Omega) \times H^{m+1}(\Omega)$. In $H^{m+1}(\Omega; \mathbb{R}^3)$ we define the following quadratics forms:

$$R_{m,\Omega}(v) = \sum_{i=1}^{3} ((rot\, v)_i | (rot\, v)_i)_{m-1,\tau,\Omega} \quad \text{and} \quad D_{m,\Omega}(v) = (div\, v | div\, v)_{m-1,\tau,\Omega},$$

for $v \in H^{m+1}(\Omega; \mathbb{R}^3)$. Let us denote by $\mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$, for $\ell = 0, 1, 2, 3, 4$, the following spaces:

$$\begin{aligned}
\mathcal{H}^m_0(\Omega; \mathbb{R}^3) &= H^{m+1}(\Omega; \mathbb{R}^3) = H^{m+1}(\Omega) \times H^{m+1}(\Omega) \times H^{m+1}(\Omega),\\
\mathcal{H}^m_1(\Omega; \mathbb{R}^3) &= \{v \in \mathcal{H}^m_0(\Omega; \mathbb{R}^3) | R_{m,\Omega}(v) = 0\},\\
\mathcal{H}^m_2(\Omega; \mathbb{R}^3) &= \{v \in \mathcal{H}^m_0(\Omega; \mathbb{R}^3) | D_{m,\Omega}(v) = 0\},\\
\mathcal{H}^m_3(\Omega; \mathbb{R}^3) &= \{v \in \mathcal{H}^m_0(\Omega; \mathbb{R}^3) | rot\, v = 0\},\\
\mathcal{H}^m_4(\Omega; \mathbb{R}^3) &= \{v \in \mathcal{H}^m_0(\Omega; \mathbb{R}^3) | div\, v = 0\}.
\end{aligned} \tag{5.1}$$

We also denote $\mathcal{V}^m_0(\mathbb{R}^3; \mathbb{R}^3) = \mathcal{V}^m(\mathbb{R}^3; \mathbb{R}^3)$. We recall that the spaces $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$, for $\ell = 1, 2, 3, 4$, are given by (4.1).

Let $\Omega_N = \{x^N_1, \ldots, x^N_{n(N)}\}$ be a $\mathbb{P}_{m-1}$-insolvent subset of Ω. We suppose that $n(N) \geqslant d(m)$ and $\lim_{N\to+\infty} n(N) = +\infty$. For any element $v \in \mathcal{V}_\ell(\mathbb{R}^3; \mathbb{R}^3)$ (or $v \in \mathcal{H}_\ell(\Omega; \mathbb{R}^3)$) with $\ell = 0, 1, 2, 3, 4$ we denote $A_N v = (v(x^N_i))_{1\leqslant i\leqslant n(N)}$.

For any $v \in \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$ with $\ell = 0, 1, 2, 3, 4$ and $\varepsilon \geqslant 0$, we denote by $S^\varepsilon_{N,\ell} v$ the solution of the problem $\mathcal{P}^{(\ell)}_\varepsilon(Z_N)$ given by (3.1) for $\ell = 0$ and given by (4.3) for $\ell = 1, 2, 3, 4$ where $Z_N = (v(x^N_i))_{1\leqslant i\leqslant n(N)}$. By using the general spline theory (see [4]), we can obviously verify that

(1) $S^\varepsilon_{N,\ell}$ is a projector : $S^\varepsilon_{N,\ell} S^\varepsilon_{N,\ell} = S^\varepsilon_{N,\ell}$,
(2) $S^\varepsilon_{N,\ell} S^0_{N,\ell} = S^0_{N,\ell}$ and $S^0_{N,\ell} S^\varepsilon_{N,\ell} = S^\varepsilon_{N,\ell}$.

Let R_Ω denote a restriction operator from $\mathbb{R}^3$ to Ω. The linear mapping R_Ω is continuous from $\mathcal{V}^m_0(\mathbb{R}^3; \mathbb{R}^3)$ to $\mathcal{H}^m_0(\Omega; \mathbb{R}^3)$ (see [12]) and consequently it is continuous from $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$ to $\mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$ for $\ell = 1, 2, 3, 4$. Let $\mathbb{P}_{m-1,0}(\mathbb{R}^3; \mathbb{R}^3) = \mathbb{P}_{m-1}(\mathbb{R}^3; \mathbb{R}^3)$ and let $\mathbb{P}_{m-1,\ell}(\Omega; \mathbb{R}^3)$ denote the space $R_\Omega[\mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)]$ for $\ell = 0, 1, 2, 3, 4$.

In the remainder of this section the subscript ℓ will be equal to 0, 1, 2, 3 or 4.

Definition 5.1. The bounded open set Ω has the $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$-extension property if there exists a continuous linear application E_ℓ from $\mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$ to $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$ such that $R_\Omega E_\ell u =$

u for any element $u \in \mathcal{H}_\ell^m(\Omega; \mathbb{R}^3)$. In this case E_ℓ is called an extension from $\mathcal{H}_\ell^m(\Omega; \mathbb{R}^3)$ to $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$.

To obtain sufficient conditions on a bounded open set Ω to have the $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$-extension property we can see Dautry and Lions [7], Něcas [12] and Adams [1]. The conditions are, in general, regarding the regularity of the boundary $\partial\Omega$ of Ω. For example, if Ω is a bounded domain satisfying the Lipschitz boundary condition, then by using the Stein extension Theorem [1, p. 154] we obtain that Ω has the $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$-extension property. We can also see Theorem 3.10 in [12, p. 80].

Proposition 5.1. *If the bounded open set Ω has the $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$-extension property, then for all $v \in \mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$, the minimal approximation problem*

$$\mathcal{P}_{\Omega,\ell}(v): \quad \min_{w \in \mathcal{C}_{\Omega,\ell}(v)} (\rho D_m(w) + R_m(w)), \tag{5.2}$$

where $\mathcal{C}_{\Omega,\ell}(v) = \{w \in \mathcal{V}_\ell^m(\mathbb{R}^3) \mid R_\Omega w = R_\Omega v\}$ admits a unique solution $S_{\Omega,\ell}v$ in $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$. In particular, if $R_\Omega v \in \mathbb{P}_{m-1,\ell}(\Omega; \mathbb{R}^3)$, then $S_{\Omega,\ell}v \in \mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)$.

Proof. The linear mapping $R_\Omega : \mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3) \to \mathcal{H}_\ell^m(\Omega; \mathbb{R}^3)$ is continuous (see [12]) and satisfies the following properties:

(1) $R_\Omega(\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)) = \mathcal{H}_\ell^m(\Omega; \mathbb{R}^3)$, which results from the $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$-extension hypothesis.
(2) $Ker(R_\Omega) \cap Ker(T_\ell) \subseteq Ker(R_\Omega) \cap \mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3) = \{0\}$.
(3) $Ker(R_\Omega) + Ker(T_\ell)$ is closed since $Ker(T_\ell)$ is a finite-dimensional vector subspace.

The existence and uniqueness of the solution of problem (5.2) result from the classical spline theory.

Let us suppose that $R_\Omega v = p \in \mathbb{P}_{m-1,\ell}(\Omega; \mathbb{R}^3)$ and let $\widetilde{p}$ be an extension of p to $\mathbb{R}^3$. It is clear that $\widetilde{p} \in \mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)$ and

$$\text{(i)} \quad R_\Omega \widetilde{p} = R_\Omega v \quad \text{(ii)} \quad \rho D_{m,\tau}(\widetilde{p}) + R_{m,\tau}(\widetilde{p}) = 0.$$

It follows that $\widetilde{p} = S_{\Omega,\ell}v \in \mathbb{P}_{m-1,\ell}(\mathbb{R}^3; \mathbb{R}^3)$. □

Remark 5.1. We can easily show that $S_{\Omega,\ell}$ is a projector on $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$ and satisfies the following properties:

(1) $S_{N,\ell}^\varepsilon S_{\Omega,\ell} = S_{\Omega,\ell}, \forall \varepsilon \geqslant 0$.
(2) $S_{\Omega,\ell} E_\ell^{(1)} = S_{\Omega,k} E_\ell^{(2)}$ for all extensions $E_\ell^{(1)}$ and $E_\ell^{(2)}$.
(3) $J(S_{\Omega,\ell} E_\ell u) \leqslant J(E_\ell u)$ for all $u \in \mathcal{H}_\ell^m(\Omega; \mathbb{R}^3)$.
(4) $S_{\Omega,\ell} E_\ell$ is an extension.

The element $S_{\Omega,\ell} E_\ell u$ is called the minimal $\mathcal{V}_\ell^m(\mathbb{R}^3; \mathbb{R}^3)$-extension of u relative to the semi-norm $w \to \sqrt{\rho D_{m,\tau}(w) + R_{m,\tau}(w)}$.

Now we will study the internal convergence result of the interpolating tension splines, i.e. in $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$. Let us first consider the following assertions:

(A1) For all $v \in \mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$, $\lim_{N\to+\infty} S^0_{N,\ell}v = S_{\Omega,\ell}v$ in $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$.

(A2) $Ker R_\Omega = \bigcap_{N\in\mathbb{N}} Ker A_N$.

(A3) $\Omega_N = \{x_1^N,\dots,x_{n(N)}^N\} \subset \Omega_{N+1} = \{x_1^{N+1},\dots,x_{n(N+1)}^{N+1}\} \subset \Omega$.

Proposition 5.2. *We suppose that the open set Ω has the $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$-extension property. Then we have the following implications*:

(i) (A1) *implies* (A2).

(ii) (A2) *and* (A3) *imply* (A1).

Proof. (i) (A1) $\Rightarrow$ (A2).

It is clear that $Ker R_\Omega \subset \cap_{N\in\mathbb{N}} Ker A_N$. For all $v \in \cap_{N\in\mathbb{N}} Ker A_N$ we have $A_N v = 0$, and consequently $S^0_{N,\ell}v = 0$, $\forall N \in \mathbb{N}$. From assertion (A1) we deduce that $0 = \lim_{N\to+\infty} S^0_{N,\ell}v = S_{\Omega,\ell}v$ and $0 = R_\Omega S_{\Omega,\ell}v = R_\Omega v$ which means that $v \in Ker R_\Omega$.

(ii) ((A2) and (A3))$\Rightarrow$(A1).

Since $A_N S^0_{N,\ell}v = A_N S_{\Omega,\ell}v$ then

$$J_{m,\tau,\rho}(S^0_{N,\ell}v) \leqslant J_{m,\tau,\rho}(S_{N+1}v) \leqslant J_{m,\tau,\rho}(S_{\Omega,\ell}v) \tag{5.3}$$

and

$$\left\| A_0 S^0_{N,\ell}v \right\|^2_{\mathbb{R}^{n(0)}} + J_{m,\tau,\rho}(S^0_{N,\ell}v) \leqslant \left\| A_0 S_{\Omega,\ell}v \right\|^2_{\mathbb{R}^{n(0)}} + J_{m,\tau,\rho}(S_{\Omega,\ell}v) \quad \forall N \in \mathbb{N}.$$

We recall that $J_{m,\tau,\rho}(v)$ is given by $J_{m,\tau,\rho}(v) = \rho D_{m,\tau}(v) + R_{m,\tau}(v)$. Since the functional $\left(\|A_0 v\|^2_{\mathbb{R}^{n(0)}} + J_{m,\tau,\rho}(v)\right)^{1/2}$ is a norm on $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$, equivalent to the usual norm $||v||_{\mathcal{V}^m}$, then $(S^0_{N,\ell}v)_{N\in\mathbb{N}}$ is bounded in $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$. Thus, we can find a subsequence $(S^0_{N_k,\ell}v)_{k\in\mathbb{N}}$ which weakly converges to w in $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$. From inequality (5.3) we obtain

$$J_{m,\tau,\rho}(w) \leqslant \liminf_{k\to+\infty} J_{m,\tau,\rho}(S^0_{N_k,\ell}v) \leqslant J_{m,\tau,\rho}(S_{\Omega,\ell}v). \tag{5.4}$$

For any fixed $N \in \mathbb{N}$ and $N_k \geqslant N$, we have $A_N S_{\Omega,\ell}v = A_N S^0_{N,\ell}v = A_N S^0_{N_k,\ell}v$ and $A_N(w - S_{\Omega,\ell}v) = \lim_{k\to+\infty} A_N(w - S^0_{N_k,\ell}v) = 0$ in $\mathbb{R}^{3n(N)}$, which means that $A_N w = A_N S_{\Omega,\ell}v$, $\forall N \in \mathbb{N}$, and $w - S_{\Omega,\ell}v \in \cap_{N\in\mathbb{N}} Ker A_N = Ker R_\Omega$.

Since $R_\Omega w = R_\Omega S_{\Omega,\ell}v$, using inequality (5.4) and the uniqueness of solution $S_{\Omega,\ell}v$ of problem (5.2), we conclude that $w = S_{\Omega,\ell}v$.

In the same manner, we can show that every weakly convergent subsequence of $(S^0_{N,\ell}v)_{N\in\mathbb{N}}$ is necessarily weakly convergent to $S_{\Omega,\ell}v$, and consequently the sequence $(S^0_{N,\ell}v)_{N\in\mathbb{N}}$ is weakly convergent to $S_{\Omega,\ell}v$. From inequalities (5.3), (5.4) and

$A_0 S_{\Omega,\ell} v = A_0 S^0_{N,\ell} v$, we obtain the norms convergence

$$\lim_{N\to+\infty} (\|A_0 S^0_{N,\ell} v\|^2_{\mathbb{R}^{n(0)}} + (\rho D_{m,\tau}(S^0_{N,\ell} v) + R_{m,\tau}(S^0_{N,\ell} v)))$$
$$= \left\|A_0 S_{\Omega,\ell} v\right\|^2_{\mathbb{R}^{n(0)}} + (\rho D_{m,\tau}(S_{\Omega,\ell} v) + R_{m,\tau}(S_{\Omega,\ell} v)).$$

The weak convergence and the norms convergence imply a strong convergence. This concludes the proof. □

We are now in a position to state the external theorem convergence of the interpolating tension splines in $\mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$.

Theorem 5.1. *For all $u \in \mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$, let $S^0_{N,\ell,\Omega} u$ be the restriction toΩ of the solution of the problem $\mathcal{P}^\ell_0(A_N u)$ in $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$, given by* (3.1) *and* (4.3). *If the following hypotheses*:

(H1) *The open set Ω has the $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$-extension property,*
(H2) $\Omega_N = \{x^N_1, \ldots, x^N_{n(N)}\} \subset \Omega_{N+1} = \{x^{N+1}_1, \ldots, x^{N+1}_{n(N+1)}\} \subset \Omega$, *and*
(H3) $\forall x \in \Omega$: $\lim_{N\to+\infty} \min_{1\leqslant i\leqslant n(N)} \| x - x^N_i \|_{\mathbb{R}^3} = 0$,

are satisfied, then we have $\lim_{N\to+\infty} S^0_{N,\ell,\Omega} u = u$ *in* $\mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$ *and consequently in* $\mathcal{C}^{m-1}(\Omega; \mathbb{R}^3)$.

Proof. The inclusion $Ker\, R_\Omega \subset \cap_{N\in\mathbb{N}} Ker\, A_N$ is a consequence of hypothesis (H2). From the imbedding Sobolev theorem (see [12]), we have

$$\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3) \hookrightarrow H^{m+1}_{loc}(\mathbb{R}^3; \mathbb{R}^3) \hookrightarrow \mathcal{C}^{m-1}(\mathbb{R}^3; \mathbb{R}^3);$$

we deduce that every function $v \in \cap_{N\in\mathbb{N}} Ker\, A_N$ is continuous on Ω, i.e.,

$$\forall x \in \Omega, \forall \varepsilon > 0, \exists \eta_\varepsilon > 0, \forall y \in \Omega, \| x - y \|_{\mathbb{R}^3} < \eta_\varepsilon \Rightarrow \|v(x) - v(y)\|_{\mathbb{R}^3} < \varepsilon.$$

Since $\lim_{N\to+\infty} \min_{1\leqslant i\leqslant n(N)} \| x - x^N_i \|_{\mathbb{R}^3} = 0$, there exist *i* and *N* such that $\| x - x^N_i \|_{\mathbb{R}^3} < \eta_\varepsilon$; then $\forall \varepsilon > 0$, $\|v(x)\|_{\mathbb{R}^3} < \varepsilon$, and consequently $v(x) = 0$, $\forall x \in \Omega$ i.e. $v \in Ker\, R_\Omega$ and $\cap_{N\in\mathbb{N}} Ker\, A_N \subset Ker\, R_\Omega$.

For all $u \in \mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$, let $v = E_\ell u \in \mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$ where E_ℓ is an extension from $\mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$ to $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3)$. Since $(A_N u)_{N\in\mathbb{N}} = (A_N v)_{N\in\mathbb{N}}$ and hypotheses (H1)–(H2)–(H3) are satisfied, it follows from Proposition 5.2 that $\lim_{N\to+\infty} S^0_{N,\ell} E_\ell u = S_{\Omega,\ell} E_\ell u$ in $\mathcal{V}_\ell(\mathbb{R}^3; \mathbb{R}^3)$. Since $S^0_{N,\ell,\Omega} u = R_\Omega S^0_{N,\ell} E_\ell u$, by using the continuity of the mapping R_Ω : $\mathcal{V}^m_\ell(\mathbb{R}^3; \mathbb{R}^3) \to \mathcal{H}^m_\ell(\Omega; \mathbb{R}^3)$ we obtain

$$\lim_{N\to+\infty} S^0_{N,\ell,\Omega} u = \lim_{N\to+\infty} R_\Omega S^0_{N,\ell} E_\ell u = R_\Omega S_{\Omega,k} E_\ell u = u, \quad \text{in } \mathcal{H}^m_\ell(\Omega; \mathbb{R}^3).$$

The convergence in $\mathcal{C}^{m-1}(\Omega; \mathbb{R}^3)$ is deduced from the continuous imbedding $\mathcal{H}^m_\ell(\Omega; \mathbb{R}^3) \hookrightarrow H^{m+1}(\Omega; \mathbb{R}^3; \mathbb{R}^3)$ and the imbedding Sobolev theorem (see [12]) $H^{m+1}(\Omega; \mathbb{R}^3) \hookrightarrow \mathcal{C}^{m-1}(\Omega; \mathbb{R}^3)$. □

Now we state the internal convergence result of the smoothing vectorial tension splines. We consider the following assertions:

(A1′) For all $v \in \mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$ and $\varepsilon > 0$, $\lim_{N\to+\infty} S_{N,\ell}^\varepsilon v = S_{\Omega,\ell} v$ in $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$.

(A2′) For all $v \in \mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$, if $R_\Omega v \in \mathcal{H}_\ell^m(\Omega;\mathbb{R}^3)\setminus \mathbb{P}_{m-1,\ell}(\Omega;\mathbb{R}^3)$ then

$$\lim_{N\to+\infty} \sum_{i=1}^{n(N)} \| v(x_i^N) \|^2_{\mathbb{R}^{3n(N)}} = +\infty.$$

(A3′) $\Omega_N = \{x_1^N,\ldots,x_{n(N)}^N\} \subset \Omega_{N+1} = \{x_1^{N+1},\ldots,x_{n(N+1)}^{N+1}\} \subset \Omega.$

Proposition 5.3. *We suppose that the open set Ω has the $\mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$-extension property. We have the following implications*:

(i) (A1′) *implies* (A2′).
(ii) (A2′) *and* (A3′) *imply* (A1′).

Proof. (i) (A1′) ⇒ (A2′).

For all $\varepsilon > 0$, for all $N \in \mathbb{N}$ and for all $v \in \mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$ the minimization property of $S_{N,\ell}^\varepsilon v$ implies that

$$\langle T_\ell S_{N,\ell}^\varepsilon v | T_\ell S_{N,\ell}^\varepsilon v\rangle_{\mathcal{Y}_\ell} + \varepsilon \| A_N(S_{N,\ell}^\varepsilon v - S_{\Omega,\ell} v) \| \leqslant \langle T_\ell S_{\Omega,\ell} v | T_\ell S_{\Omega,\ell} v\rangle_{\mathcal{Y}_\ell}$$

Let us suppose that (A1′) is true, it follows that

$$\lim_{N\to+\infty} \langle T_\ell S_{N,\ell}^\varepsilon v | T_\ell S_{N,\ell}^\varepsilon v\rangle_{\mathcal{Y}_\ell} = \langle T_\ell S_{\Omega,\ell} v | T_\ell S_{\Omega,\ell} v\rangle_{\mathcal{Y}_\ell},$$

and consequently

$$\lim_{N\to+\infty} \| A_N(S_{N,\ell}^\varepsilon v - S_{\Omega,\ell} v) \|_{\mathbb{R}^{3n(N)}} = 0. \tag{5.5}$$

If (A2′) is false, then there exists $u \in \mathcal{V}_\ell(\mathbb{R}^3;\mathbb{R}^3)$ and there exists $M > 0$ such that $R_\Omega v \in \mathcal{H}_\ell^m(\Omega;\mathbb{R}^3)\setminus \mathbb{P}_{m-1,\ell}(\Omega;\mathbb{R}^3)$ and $\forall N \in \mathbb{N}$, $||A_N u||_{\mathbb{R}^{3n(N)}} \leqslant M$. By using the Cauchy–Schwarz inequality, we have

$$\left| \left\langle A_N(S_{N,\ell}^\varepsilon v - S_{\Omega,\ell} v) \mid A_N u\right\rangle_{\mathbb{R}^{3n(N)}} \right| \leqslant M \| A_N(S_{N,\ell}^\varepsilon v - S_{\Omega,\ell} v) \|_{\mathbb{R}^{3n(N)}} .$$

According to relation (5.5), we obtain that

$$\lim_{N\to+\infty} \left| \left\langle A_N(S_{N,\ell}^\varepsilon v - S_{\Omega,\ell} v) \mid A_N u\right\rangle_{\mathbb{R}^{3n(N)}} \right| = 0.$$

Since $A_N(S_{\Omega,\ell} v) = A_N v$ for all $v \in \mathcal{V}_\ell^m(\mathbb{R}^3;\mathbb{R}^3)$ and by using the characterization equation (4.7), we obtain

$$\left\langle T_\ell S_{N,\ell}^\varepsilon v | T_\ell S_{\Omega,\ell} u\right\rangle_{\mathcal{Y}_\ell} + \varepsilon\left\langle A_N(S_{N,\ell}^\varepsilon v - S_{\Omega,\ell} v) | A_N u\right\rangle_{\mathbb{R}^{3n(N)}} = 0.$$

We deduce that $\lim_{N\to+\infty}\langle T_\ell S^\varepsilon_{N,\ell}v|T_\ell S_{\Omega,\ell}u\rangle_{\mathcal{Y}_\ell} = 0$ for all $v \in \mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$. Since $S^\varepsilon_{N,\ell} S_{\Omega,\ell}u = S_{\Omega,\ell}u$, we obtain in particular for $v = S_{\Omega,\ell}u$ that

$$0 = \lim_{N\to+\infty}\langle T_\ell S^\varepsilon_{N,\ell}S_{\Omega,\ell}u|T_\ell S_{\Omega,\ell}u\rangle_{\mathcal{Y}_\ell} = \langle T_\ell S_{\Omega,\ell}u|T_\ell S_{\Omega,\ell}u\rangle_{\mathcal{Y}_\ell} = ||T_\ell S_{\Omega,\ell}u||^2_{\mathcal{Y}_\ell}.$$

Then $S_{\Omega,\ell}u \in Ker(T_\ell) = \mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3)$ and consequently $R_\Omega u = R_\Omega S_{\Omega,\ell}u$ belongs to $\mathbb{P}_{m-1,\ell}(\Omega;\mathbb{R}^3)$, which is impossible.

(ii) ((A2′) and (A3′))⇒ (A1′).

Let $J^\varepsilon_N(u) = \rho D_{m,\tau}(u) + R_{m,\tau}(u) + \varepsilon \parallel A_N u \parallel^2_{\mathbb{R}^{3n(N)}}$. From assertion (A3′) we obtain

$$J^\varepsilon_p(u) \leqslant J^\varepsilon_N(u), \quad \forall p \leqslant N. \tag{5.6}$$

For $u = S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v$ with $v \in \mathcal{V}_\ell(\mathbb{R}^3;\mathbb{R}^3)$, inequality (5.6) becomes

$$J^\varepsilon_p(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v) \leqslant J^\varepsilon_N(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v) \quad \forall p \leqslant N, \tag{5.7}$$

and the characterization equation (4.7) becomes

$$\langle T_\ell S^\varepsilon_{N,\ell}v|T_\ell(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v)\rangle_{\mathcal{Y}_\ell} = -\varepsilon||A_N(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v)||^2_{\mathbb{R}^{3n(N)}}.$$

Furthermore, since

$$\begin{aligned}||T_\ell(S^\varepsilon_{N,k}v - S_{\Omega,k}v)||^2_{\mathcal{Y}_\ell} = {}& \langle T_\ell S^\varepsilon_{N,\ell}v|T_\ell(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v)\rangle_{\mathcal{Y}_\ell}\\ &-\langle T_\ell S_{\Omega,\ell}v|T_\ell(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v)\rangle_{\mathcal{Y}_\ell},\end{aligned}$$

it follows that

$$\begin{aligned}J^\varepsilon_N(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v) &= -\langle T_\ell S_{\Omega,\ell}v|T_\ell(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v)\rangle_{\mathcal{Y}_\ell}\\ &= -\langle T_\ell S_{\Omega,\ell}v|T_\ell S^\varepsilon_{N,\ell}v\rangle_{\mathcal{Y}_\ell} + ||T_\ell S_{\Omega,\ell}v||^2_{\mathcal{Y}_\ell}.\end{aligned} \tag{5.8}$$

Using the minimization property of $S^\varepsilon_{N,\ell}v$ and the fact that $A_N v = A_N S_{\Omega,\ell}v$ for all $N \in \mathbb{N}$, we obtain

$$||T_\ell S^\varepsilon_{N,\ell}v||^2_{\mathcal{Y}_\ell} + \varepsilon||A_N(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v)||^2_{\mathbb{R}^{3n(N)}} \leqslant ||T_\ell S_{\Omega,\ell}v||^2_{\mathcal{Y}_\ell}.$$

Thus, from (5.8) by using Cauchy–Schwarz inequality, we obtain

$$J^\varepsilon_N(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v) \leqslant 2||T_\ell S_{\Omega,\ell}v||^2_{\mathcal{Y}_\ell} \quad \forall N \in \mathbb{N}. \tag{5.9}$$

From Eqs. (5.6) and (5.9), we deduce that

$$J^\varepsilon_0(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v) \leqslant 2||T_\ell S_{\Omega,\ell}v||^2_{\mathcal{Y}_\ell} \quad \forall N \in \mathbb{N}. \tag{5.10}$$

Since $\sqrt{J^\varepsilon_0(\,.\,)}$ is a norm on $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$ equivalent to the usual norm $||\,.\,||_{\mathcal{V}^m}$, then $(S^\varepsilon_{N,\ell}v - S_{\Omega,\ell}v)_{N\in\mathbb{N}}$ is a bounded sequence in $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$. Thus, there is a subsequence $(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v)_{k\in\mathbb{N}}$ which weakly converges in $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$ to an element $u \in \mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$. Then

$$J^\varepsilon_N(u) \leqslant \liminf_{k\to+\infty} J^\varepsilon_N(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v) \leqslant 2||T_\ell S_{\Omega,\ell}v||^2_{\mathcal{Y}_\ell} \quad \forall N \in \mathbb{N},$$

and consequently the sequence of positive real numbers $(||A_N u||_{\mathbb{R}^{3n(N)}})_{N\in\mathbb{N}}$ is bounded. It follows from assertion $(A2')$ that $R_\Omega u \in \mathbb{P}_{m-1,\ell}(\Omega;\mathbb{R}^3)$ and consequently $S_{\Omega,k}u \in \mathbb{P}_{m-1,\ell}(\mathbb{R}^3;\mathbb{R}^3) = Ker\ T_\ell$ (see Proposition 5.1). By using Eqs. (5.7) and (5.8) we obtain

$$J_0^\varepsilon(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v) \leqslant J^\varepsilon_{N_k}(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v) = -\left\langle T_\ell S_{\Omega,\ell}v | T_\ell(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v)\right\rangle_{\mathcal{Y}_\ell}.$$

Thus

$$0 \leqslant \lim_{k\to+\infty} J_0^\varepsilon(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v) \leqslant -\lim_{k\to+\infty}\left\langle T_\ell S_{\Omega,\ell}v | T_\ell(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v)\right\rangle_{\mathcal{Y}_\ell}. \quad (5.11)$$

It is well known that the spline $S_{\Omega,\ell}v$ satisfies the characterization equation

$$\left\langle T_\ell S_{\Omega,\ell}v | T_\ell w\right\rangle_{\mathcal{Y}_\ell} = 0 \quad \forall w \in Ker\ R_\Omega. \quad (5.12)$$

By taking $w = u - S_{\Omega,\ell}u$ in Eq. (5.12), we obtain

$$\left\langle T_\ell S_{\Omega,\ell}v | T_\ell u\right\rangle_{\mathcal{Y}_\ell} = \left\langle T_\ell S_{\Omega,\ell}v | T_\ell S_{\Omega,\ell}u\right\rangle_{\mathcal{Y}_\ell} = 0.$$

The last equation is equal to zero, because $S_{\Omega,k}u \in Ker\ T_\ell$. The weak convergence of the subsequence $(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v)_{k\in\mathbb{N}}$, to u gives

$$\lim_{k\to+\infty}\left\langle T_\ell S_{\Omega,\ell}v | T_\ell(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v)\right\rangle_{\mathcal{Y}_\ell} = \left\langle T_\ell S_{\Omega,\ell}v | T_\ell u\right\rangle_{\mathcal{Y}_\ell} = 0.$$

From inequality (5.11), we obtain $\lim_{k\to+\infty} J_0^\varepsilon(S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v) = 0$. It follows that $\lim_{k\to+\infty} ||S^\varepsilon_{N_k,\ell}v - S_{\Omega,\ell}v||_{\mathcal{V}^m} = 0$, i.e., the sequence $(S^\varepsilon_{N_k,\ell}v)_{k\in\mathbb{N}}$ converges to $S_{\Omega,\ell}v$ in $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$. In the same manner, we can show that every convergent subsequence of $(S^\varepsilon_{N,\ell}v)_{N\in\mathbb{N}}$ is necessarily convergent to $S_{\Omega,\ell}v$ and consequently the sequence $(S^\varepsilon_{N,\ell}v)_{N\in\mathbb{N}}$ is convergent to $S_{\Omega,\ell}v$. This concludes the proof. □

Remark 5.2. (1) If assertions (A2′) and (A3′) of Proposition 5.3 are satisfied, then can obviously be verified that $Ker R_\Omega = \bigcap_{N\in\mathbb{N}} Ker A_N$. From Proposition 5.2 we deduce that $\lim_{N\to+\infty} S^0_{N,\ell}v = S_{\Omega,\ell}v$ in $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$.

(2) Propositions 5.2 and 5.3 were proved by Atteia [4], in the particular case where $\Omega = \mathbb{R}^3$. In this case $Ker R_\Omega = \{0_{\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)}\}$ and $S_{\Omega,\ell}$ is the identity operator of $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$.

Now we state the theorem relative to the convergence in $\mathcal{H}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$ of smoothing tension splines.

Theorem 5.2. *For all $u \in \mathcal{H}^m_\ell(\Omega;\mathbb{R}^3)$, let $S^\varepsilon_{N,\ell,\Omega}u$ be the restriction to Ω of the solution of the problem $\mathcal{P}^\varepsilon_\ell(A_N u)$ in $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$, given by* (3.1) *and* (4.3). *If the following hypotheses*:

(H1) *The open set Ω has the $\mathcal{V}^m_\ell(\mathbb{R}^3;\mathbb{R}^3)$-extension property,*
(H2) $\Omega_N = \{x^N_1, \ldots, x^N_{n(N)}\} \subset \Omega_{N+1} = \{x^{N+1}_1, \ldots, x^{N+1}_{n(N+1)}\} \subset \Omega,$

(H3) *For all* $v \in \mathcal{V}_\ell^m(\mathbb{R}^3)$, *if* $R_\Omega v \in \mathcal{H}_\ell^m(\Omega;\mathbb{R}^3) \setminus \mathbb{P}_{m-1,\ell}(\Omega;\mathbb{R}^3)$ *then*

$$\lim_{N\to+\infty} \sum_{i=1}^{n(N)} \| v(x_i^N) \|_{\mathbb{R}^3}^2 = +\infty,$$

are satisfied, then for all $\varepsilon \geqslant 0$, *we have* $\lim_{N\to+\infty} S_{N,\ell,\Omega}^{\varepsilon} u = u$ in $\mathcal{H}_\ell^m(\Omega;\mathbb{R}^3)$ *and consequently in* $\mathcal{C}^{m-1}(\Omega;\mathbb{R}^3)$.

Proof. It is analoguous to the proof of Theorem 5.1. □

Acknowledgments

The authors thank the anonymous referees for their useful reading which helped to improve the presentation of this paper.

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